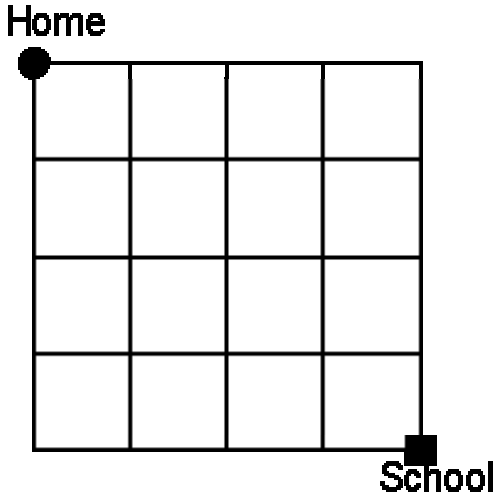
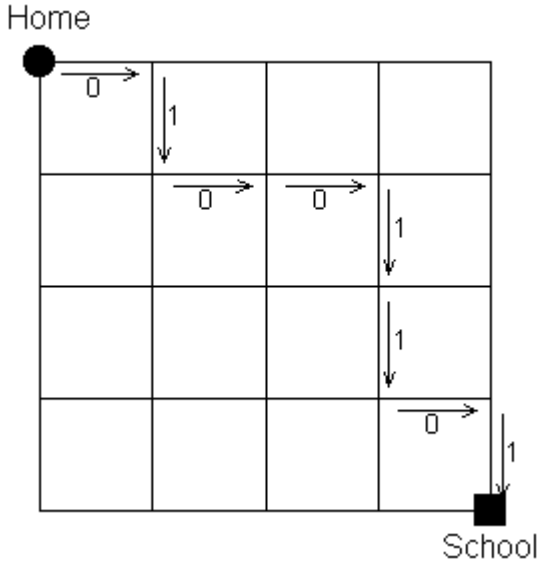
Lesson 3
Counting Things

LOUIS'S PUZZLE



Louis lives four blocks north and four blocks west of school. How many different routes can he take to school (always heading south or east)? An example of one route is shown below. There are many such paths. How can we keep track of them in an orderly way so we counted each path once and only once?



Let movement one block east be represented by a "0," and let movement one block south be represented by a "1." Then each path can be coded as a pattern of four 0s and four 1s. For example, the path shown above would be coded as 01001101. Of the eight numbers, exactly four of them must be 1s (and the rest 0s). How many different ways of choosing four things out of eight are there? Answer: the combination of 8 things taken four at a time is the product of the four numbers counting down from 8 divided by the product of the four numbers counting up from 1:

$$C_4^8 = \frac{8\times7\times6\times5}{1\times2\times3\times4} = \frac{1680}{24} = 70$$

In general, the combination of *n* things taken *m* at a time is the product of the *m* numbers counting down from *n* divided by the product of the *m* numbers counting up from 1:

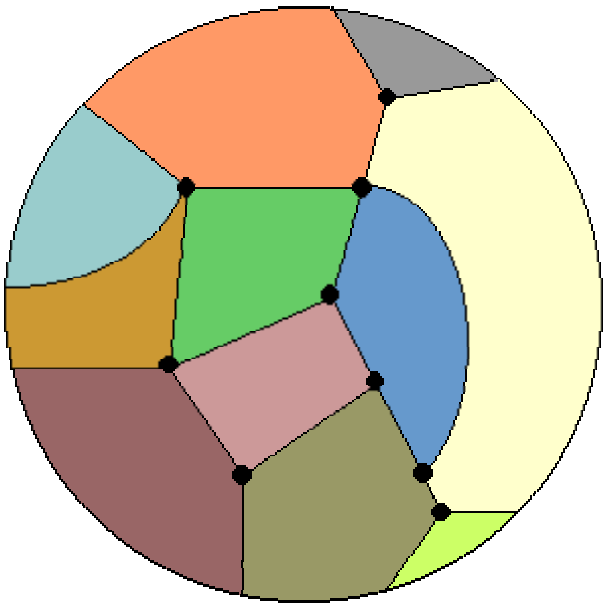
$$C_m^n = \frac{n\times(n-1)\times(n-2)\times\cdots\times(n-m+1)}{1\times2\times3\times\cdots\times m}$$

If you don't know the equation for *n* things taken *m* at a time, you can at least list all of the combinations in some orderly way and count them. Here are the 70 combinations of 8 things taken 4 at a time:

11110000	11000011	10010110	01100110	00111010
11101000	10111000	10010101	01100101	00111001
11100100	10110100	10010011	01100011	00110110
11100010	10110010	10001110	01011100	00110101
11100001	10110001	10001101	01011010	00110011
11011000	10101100	10001011	01011001	00101110
11010100	10101010	10000111	01010110	00101101
11010010	10101001	01111000	01010101	00101011
11010001	10100110	01110100	01010011	00100111
11001100	10100101	01110010	01001110	00011110
11001010	10100011	01110001	01001101	00011101
11001001	10011100	01101100	01001011	00011011
11000110	10011010	01101010	01000111	00010111
11000101	10011001	01101001	00111100	00001111

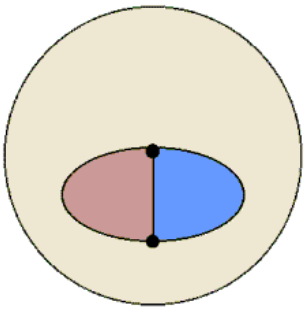
GRENZA'S PUZZLE

A planet with no bodies of water is divided into countries by lines that meet (terminate) at points. No country is an "island" (surrounded by one country). There are 37 lines and 15 points. How many countries are there?



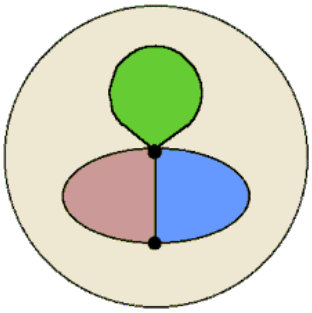
Start out with a simple case of 3 countries, 3 lines, and 2 points. (The outer circle is not a line, just the rim of the planet. The planet is all gray on the back side.)

$C = 3, L = 3, P = 2.$



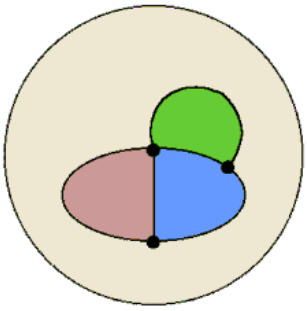
One way to add another country is to just add a line terminating on the points already there.

$C = 4, L = 4, P = 2.$



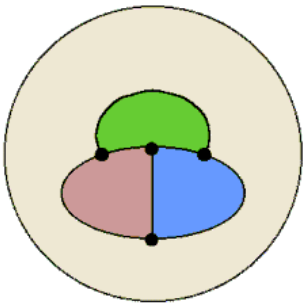
Another way to add a country to the first picture is to add a line terminating on one of the points already there, but the other end terminates at a new point that divides one to the lines already there. So there are 2 new lines and 1 new point.

$C = 4, L = 5, P = 3.$



Another way to add a country to the first picture is to add a line terminating on two new points, each new point dividing a line already there. So there are 3 new lines and 2 new points.

$C = 4, L = 6, P = 4.$



In general, the number of countries increases by the number of new lines minus the number of new points. So the equation for countries in terms of lines and points must be of the form

$$C = L - P + k,$$

where k is a constant that doesn't change with C, L, or P. We can find k by looking at any of the cases above. For the first case $C = 3$, $L = 3$, $P = 2$, and substituting into $C = L - P + k$ gives

$$3 = 3 - 2 + k$$
$$3 = 1 + k,$$

so k must be 2:

$$C = L - P + 2.$$

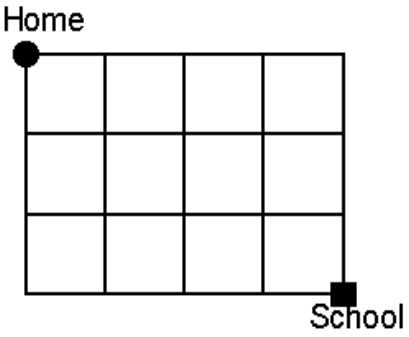
For $L = 37$ and $P = 15$ (see original puzzle), we get

$$C = 37 - 15 + 2 = 24.$$

Homework

1.

Louis lives three blocks north and four blocks west of school. How many ways can he walk to school if he always heads south or east? Calculate the answer by using the equation for n things taken m at a time. Check your answer by listing all of the patterns of 1s and 0s.



2.

Tuesday will be our last session. I'll answer questions you have about math—an area we've talked about, or something entirely new.