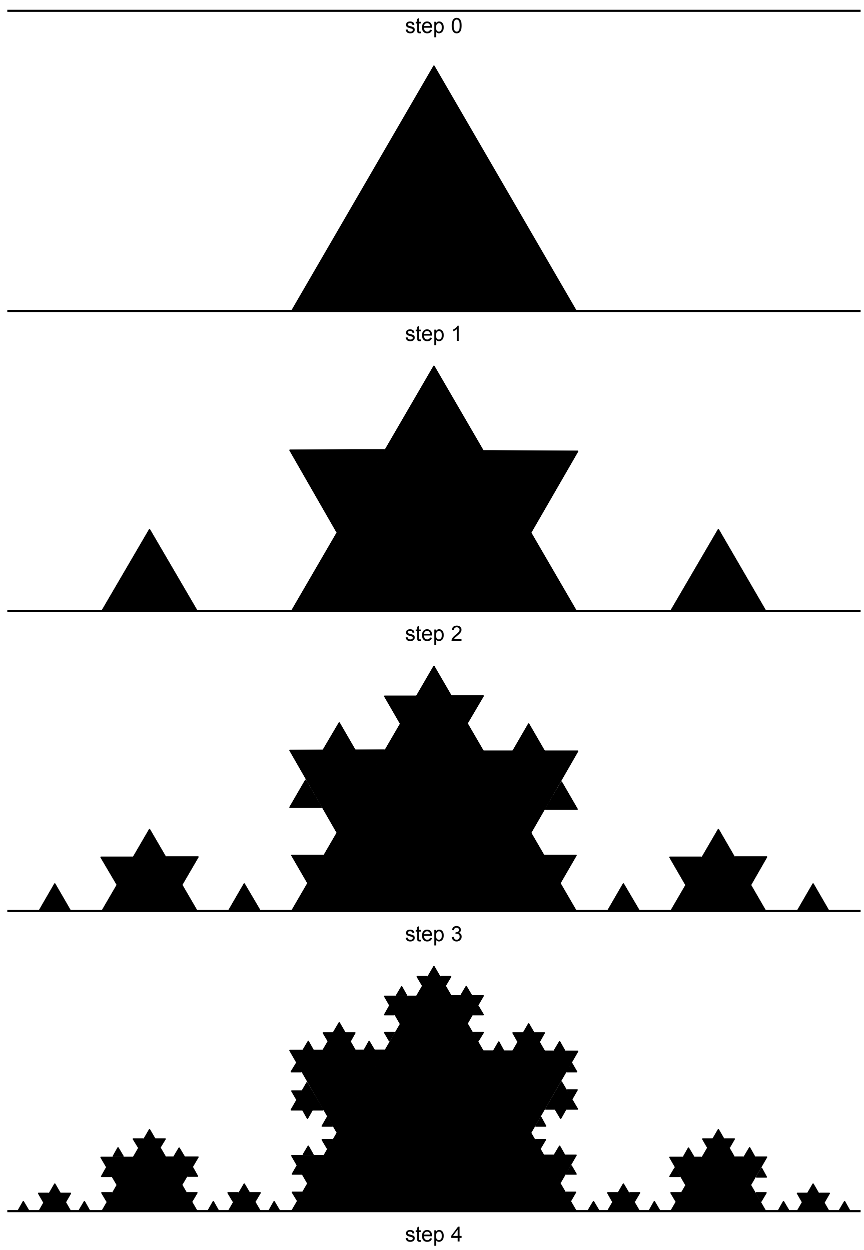
Lesson 2
Fractals and Tiles

Koch Snowflake



Step 0 is simply a line of length 1. Step 1 adds an equilateral triangle with a base one-third the length of the line set in the middle of the line. Step 2 takes four triangles one-third the size (one-ninth the area) of the first and places them in the middle of the four lines around the perimeter in Step 1. Step 3 again adds four triangles (scaled down by a factor of three) for each of the four new triangles in step 2. Step 4 adds four triangles (scaled down by a factor of three) for each of the 16 new triangles in Step 3. Etc…

As the steps go to infinity, how long does the perimeter get (the distance along the outside of all the triangles)? Step 1 takes the middle third of the line and doubles it (the two sides of the equilateral triangle replace it in the new perimeter). So the perimeter has increased by a factor $1^1/_3$, or a factor of 4/3. In Step 2 each of the four line segments in Step 1 is again increased in length by a factor of 4/3, so the whole perimeter is increased by a factor of 4/3. Thus each step increases the perimeter by a factor of 4/3.

Step 0: perimeter = 1
Step 1: perimeter = 4/3
Step 2: perimeter = $(4/3)\times(4/3) = (4/3)^2$
Step 3: perimeter = $(4/3)\times(4/3)\times(4/3) = (4/3)^3$
Step 4: perimeter = $(4/3)\times(4/3)\times(4/3)\times(4/3) = (4/3)^4$
Step n: perimeter = $(4/3)^n$
Step ∞: perimeter = $(4/3)^\infty = \infty$

So all the "fuzz" on the perimeter eventually makes the perimeter infinite.

Does the area also go to infinity? It doesn't look like it. Let the area of the triangle in Step 1 be A. In Step 2 we add four triangles, each with one-third the size (one ninth the area). So the total area for each step is

Step 0: area = 0
Step 1: area = A
Step 2: area = $A + 4\times(1/9)\times A$
Step 3: area = $A + 4\times(1/9)\times A + 16\times(1/9)\times(1/9)\times A$
$= A + 4\times(1/9)\times A + 4\times4\times(1/9)\times(1/9)\times A$
$= [1 + (4/9) + (4/9)^2]\times A$
Step 4: area = $[1 + (4/9) + (4/9)^2 + (4/9)^3]\times A$
Step n: area = $[1 + (4/9) + (4/9)^2 + \ldots + (4/9)^{n-1}]\times A$
Step ∞: area = $[1 + (4/9) + (4/9)^2 + (4/9)^3 + \ldots]\times A$

How large is $1 + (4/9) + (4/9)^2 + (4/9)^3 + \ldots$ as we take the sum to infinity? There are an infinite number of terms but the terms become infinitesimal. Do they become small quickly enough that the sum is finite? Lets look at another way of getting $1 + (4/9) + (4/9)^2 + (4/9)^3 + \ldots$ First we calculate $1/[1-(4/9)]$:

$$\frac{1}{1-(4/9)} = \frac{1}{(5/9)} = \frac{9}{5} = 1+\frac{4}{5} = 1.8$$

We can also calculate $1/[1-(4/9)]$ by long division:

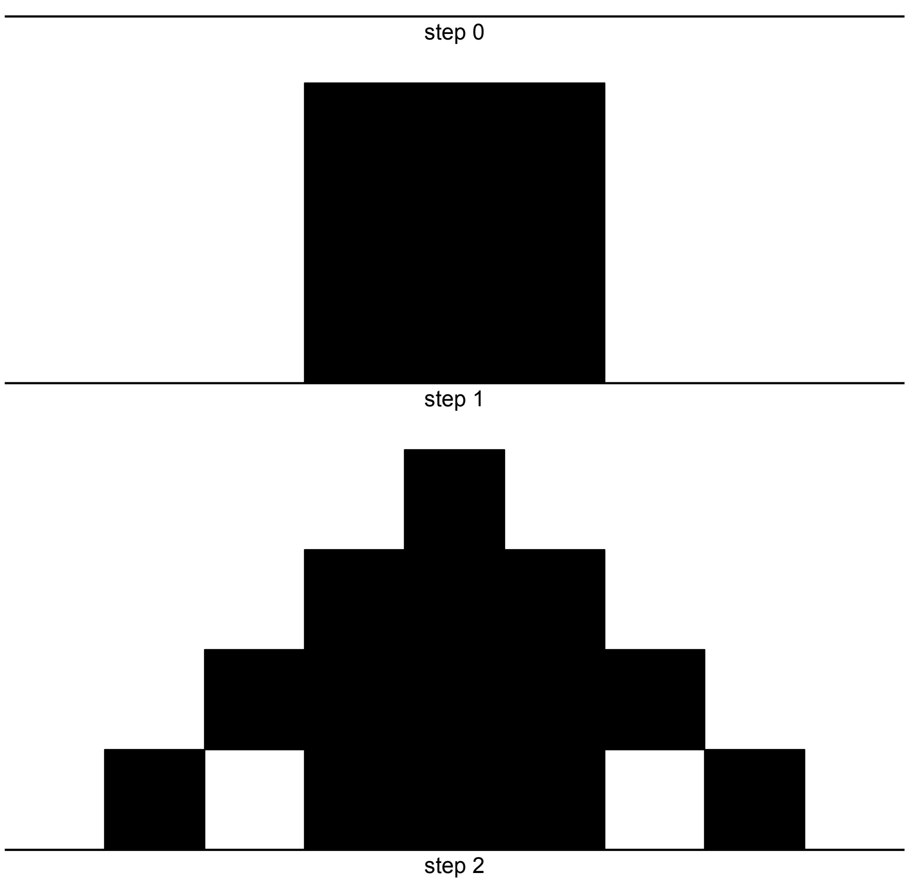
$$
\begin{array}{r}
1 + (4/9) + (4/9)^2 + (4/9)^3 + (4/9)^4 + \ldots \\
1 - (4/9) \,\overline{\big)\, 1 + 0 + 0 + 0 + 0 + \ldots} \\
\underline{1 - (4/9)} \qquad\qquad\qquad\qquad\qquad \\
(4/9) + 0 \qquad\qquad\qquad\quad \\
\underline{(4/9) - (4/9)^2} \qquad\qquad\qquad \\
(4/9)^2 + 0 \qquad\qquad \\
\underline{(4/9)^2 - (4/9)^3} \qquad\quad \\
(4/9)^3 + 0 \qquad \\
\underline{(4/9)^3 - (4/9)^4} \\
(4/9)^4
\end{array}
$$

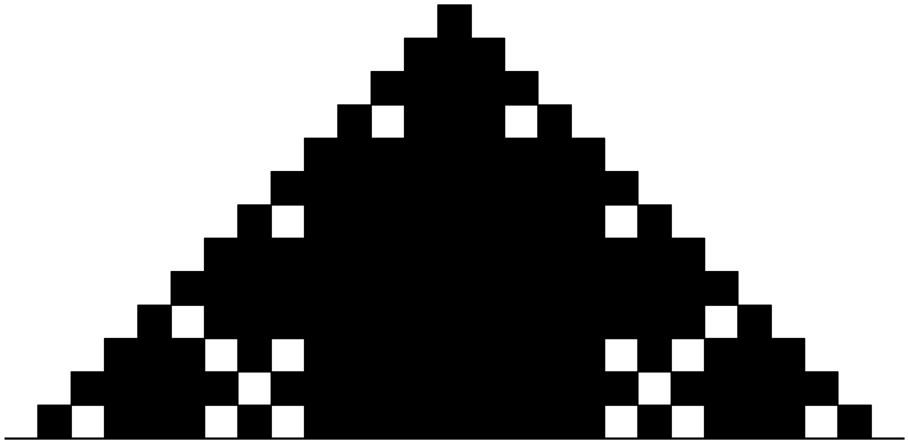
Therefore $1 + (4/9) + (4/9)^2 + (4/9)^3 + (4/9)^4 + \ldots = \dfrac{1}{1 - (4/9)} = 1.8$

So the total area at Step ∞ is $1.8 \times A$, which is finite and is an area only a little greater than the area A of the triangle in Step 1.

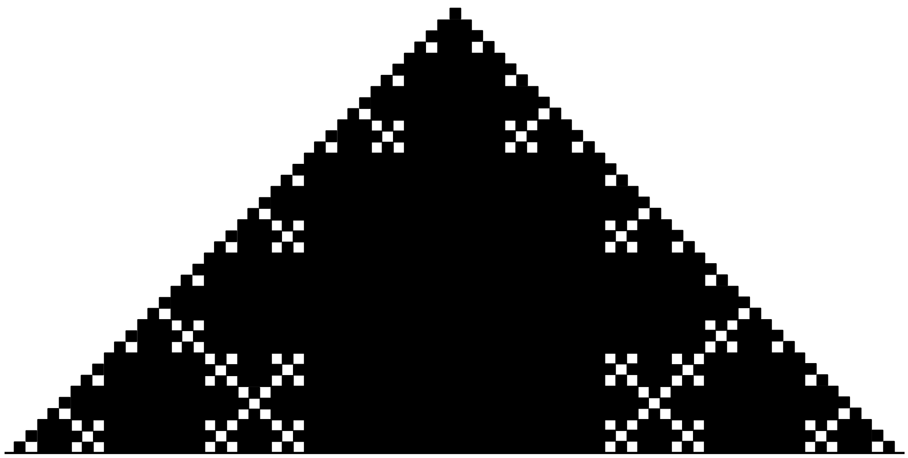
Fractal Pyramid

In a similar way, we can generate a fractal by using squares rather than triangles.

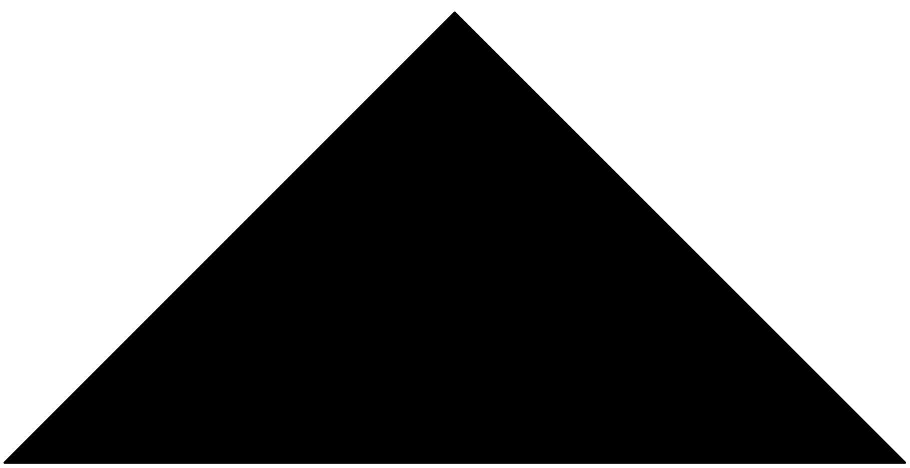




step 3



step 4



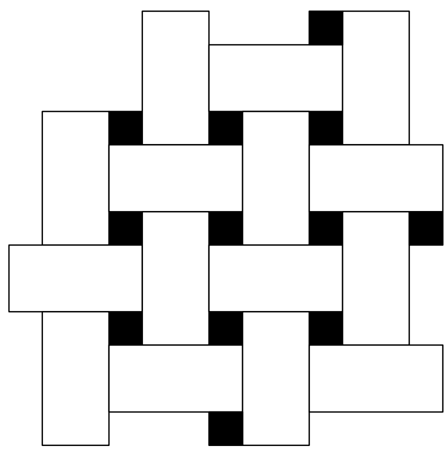
step ∞

Step 0 is simply a line of length 1. Step 1 adds a square with a base one-third the length of the line set in the middle of the line. Step 2 takes five squares one-third the size (one-ninth the area) of the first and places them in the middle of the five straight lines around the perimeter of Step 1. Step 3 again adds five squares (scaled down by a factor of three) for each of the five new squares in step 2. Step 4 adds five triangles (scaled down by a factor of three) for each of the 25 new squares in Step 3. Etc…

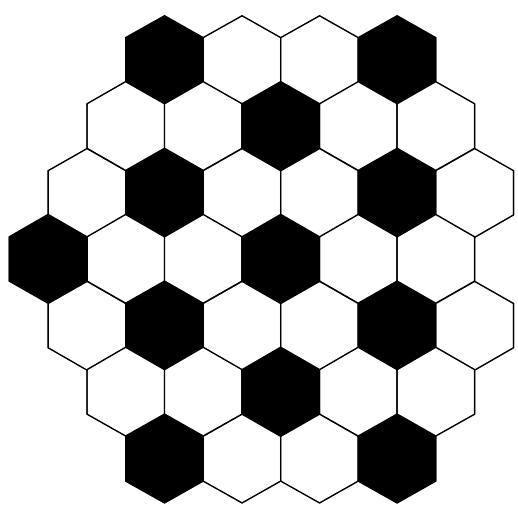
Homework Problems

1. Show that the perimeter of the Fractal Pyramid goes to infinity in Step ∞. It doesn't look like it in the picture though; the perimeter looks straight with no fuzz. But the infinite perimeter is hidden in the interior of the figure, which is like Swiss cheese with infinitesimally fine holes that can't be seen. In fact, the holes are so fine that they take up no area at all, even though there is an infinite number of then. You should be able show that the area of Step ∞ is $2.25 \times A$, based on an area of A for the square in Step 1. Use the same approach as with the Koch Snowflake.

2. A bathroom floor has a "basket weave" pattern (see below) formed by black and white tiles. As the pattern is extended to cover a large floor, are there more black tiles or white tiles? How many white tiles are there for each black tile? Make sure the white tiles you have associated with one black tile are not "claimed" by some other black tile too.



3. A bathroom floor has a "honeycomb" pattern (see below) formed by black and white tiles. As the pattern is extended to cover a large floor, are there more black tiles or white tiles? How many white tiles are there for each black tile? Make sure the white tiles you have associated with one black tile are not "claimed" by some other black tile too.



4. A soccer ball is constructed of some black pentagons and white hexagons (see below). How many pentagons are there, and how many hexagons are there. Don't mark them on the ball as you count them; use some way of associating hexagons with pentagons, as in Problems 2. and 3.

5. A bathroom floor has a "daisy field" pattern (see below) formed by black and white tiles. As the pattern is extended to cover a large floor, are there more black tiles or white tiles? How many white tiles are there for each daisy of six black tiles? Make sure the white tiles you have associated with one daisy are not "claimed" by some other daisy too.