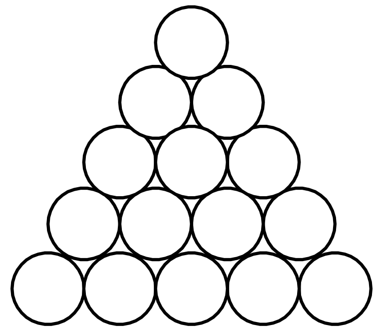
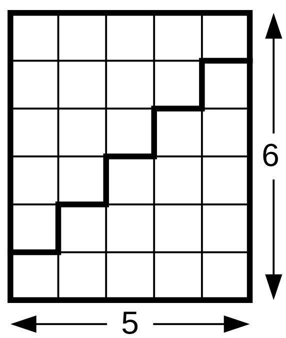
Lesson 1
Mathematics – a short-cut for counting

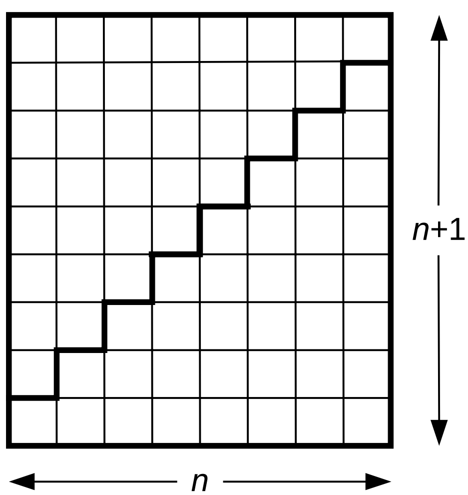
How many pool balls? 15



A quick way of adding the numbers 1 through 5: Use squares rather than balls to build the triangle, and fit a second triangle with the first to form a rectangle. Then take half the area of the rectangle.



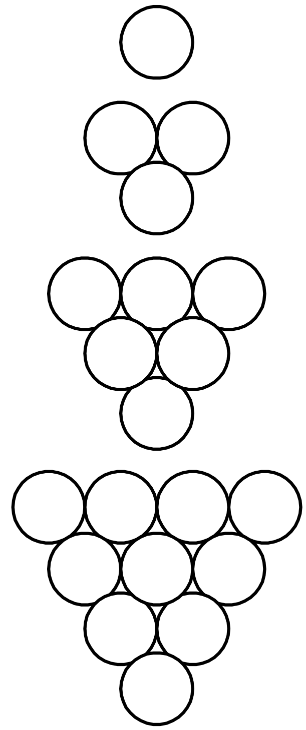
$$1+2+3+4+5 = \frac{5 \times 6}{2} = 15$$



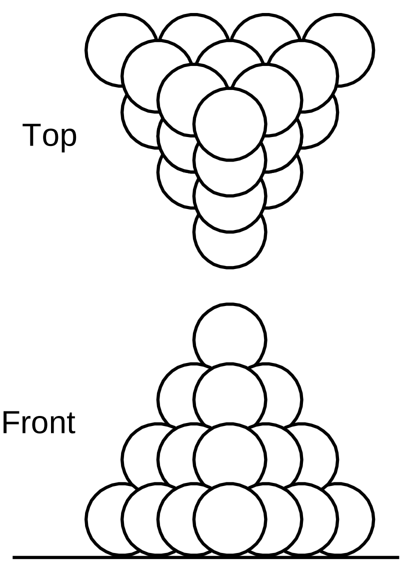
In general:

$$1+2+3+4+...+n = \frac{n \times (n+1)}{2}$$

The number 15 is an example of a *triangular number*, a number formed by adding all the numbers from 1 to some number *n*. They are called triangular numbers because that number of balls can be arranged into a triangle. The first four triangular numbers are shown at the right.

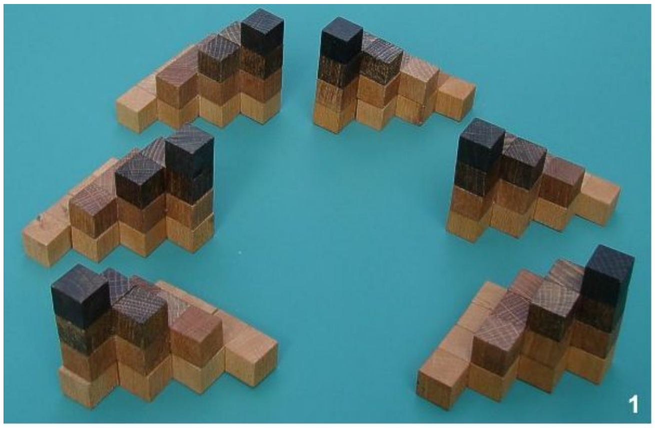


We can invent a new set of numbers called *pyramidal numbers* by adding the first *n* triangular numbers. They can be called pyramidal numbers because that number of balls can be arranged into a pyramid. The fourth pyramidal number is illustrated below. The pyramid is formed by stacking the four triangular arrangement of balls shown above. So the fourth pyramidal number is 1 + 3 + 6 + 10 = 20.



1 = 1		4th layer
1+2 = 3		3rd layer
1+2+3 = 6		2nd layer
1+2+3+4 =	10	1st layer
Total:	20	

Is there a quick way of counting the balls in a pyramid, just as there was for counting the balls in a triangle? There we put two triangles together to form a rectangle. Perhaps we can put a few pyramids together to form a rectangular solid—a "brick." (See next page.)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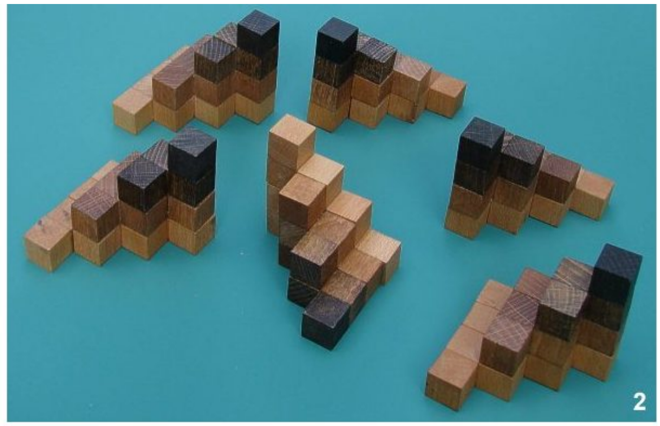
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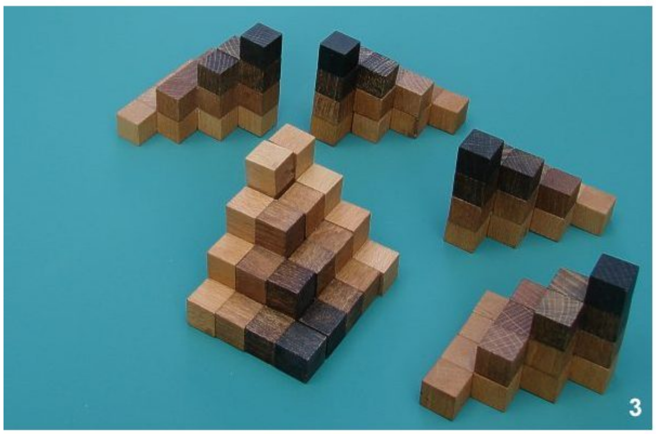
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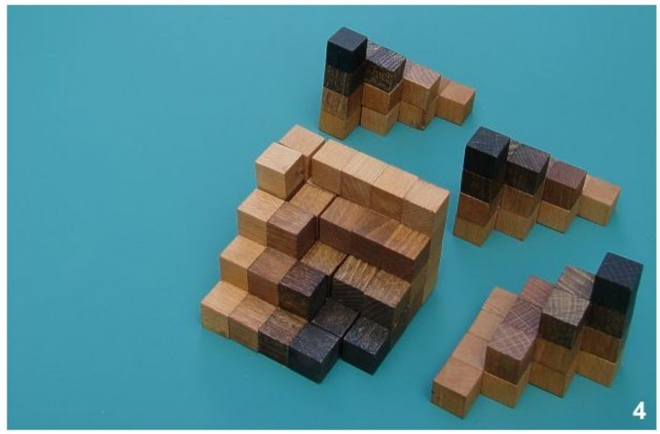
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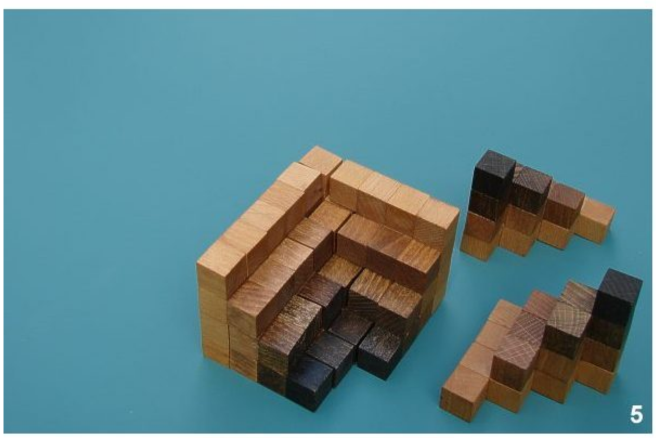
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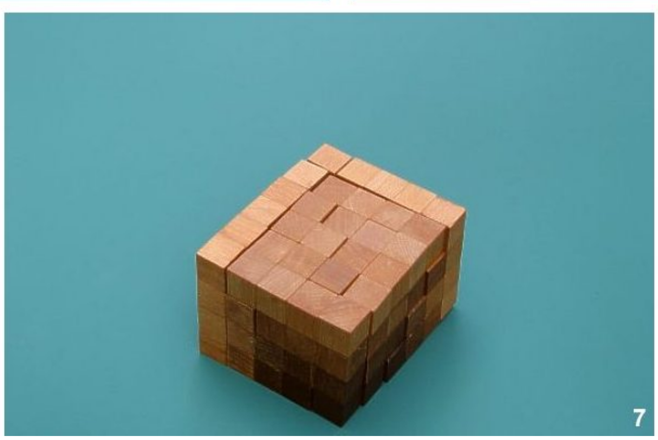
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We used cubes rather than balls to form the pyramids so they could form a square brick. It took six pyramids—each with 4 cubes on the edge—to make a brick 4 × 5 × 6. The volume of the brick is 4 × 5 × 6 = 120, and the volume (number of cubes) in each pyramid is 120/6 = 20. This agrees with 1 + 3 + 6 + 10 = 20 calculated before.

In general, the number of balls in a pyramid with *n* balls on the edge is given by

$$n = \frac{n \times (n+1) \times (n+2)}{1 \times 2 \times 3} = \frac{n \times (n+1) \times (n+2)}{6}$$

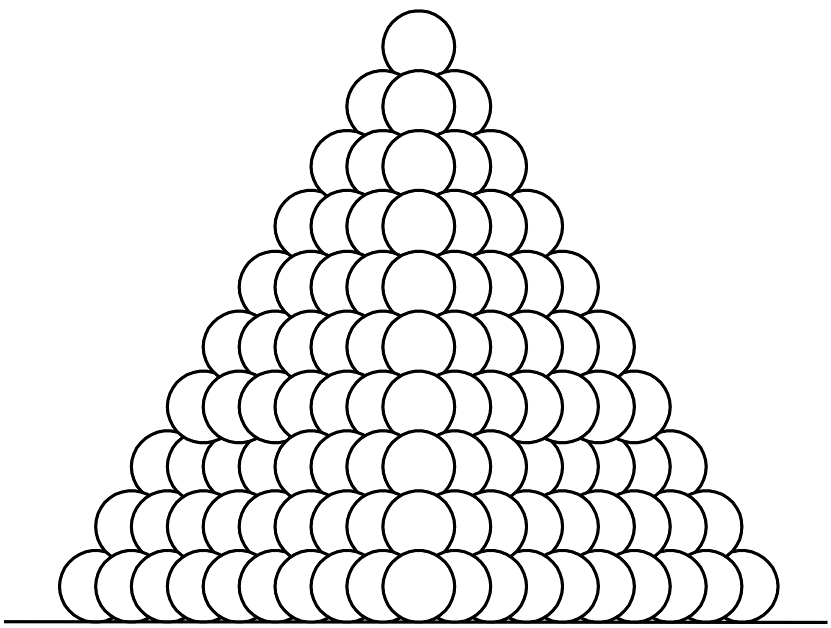
This is the *n*th pyramidal number.

Homework

(Work as many as you can for your grade level.)

1. Use the quick way to add the numbers from 1 to 256.

2. What is the 10^{th} pyramidal number? This number of balls can form a pyramid with 10 balls on each edge, as shown below.



3. What are the first five pyramidal numbers? If we lived in the fourth dimension, we could stack a 1-pyramid (single ball) on a 2-pyramid on a 3-pyramid on a 4-pyramid on a 5-pyramid to get a 5-hyperpyramid. Then we could take a few of these hyperpyramids to form a hyperbrick with four dimensions. You can't visualize this, but mathematicians extrapolate from two and three dimensions to solve problems like this. We know the *n*th triangular number (sum of the first *n* numbers) is given by

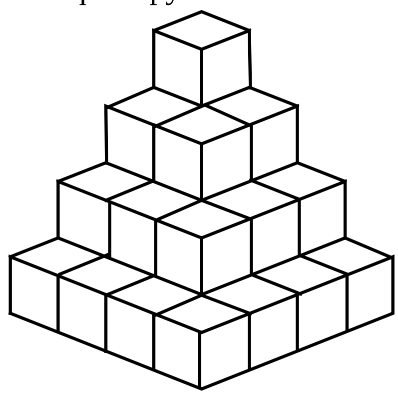
$$n = \frac{n \times (n+1)}{1 \times 2}$$

and the *n*th pyramidal number (sum of the first *n* triangular numbers) is given by

$$n = \frac{n \times (n+1) \times (n+2)}{1 \times 2 \times 3}$$

Can you guess the formula for the *n*th hyperpyramidal number? (This is the sum of the first *n* pyramidal numbers.) The denominator of the formula is the number of hyperpyramids you need to combine to get a hyperbrick. Check your formula for a couple small values of *n*.

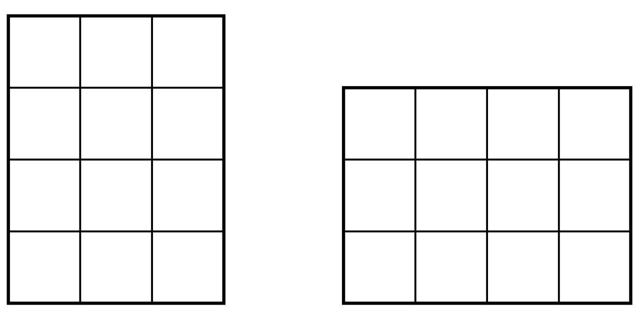
4. We can define square pyramidal numbers as the sum of the first few square numbers. For example, the fourth square pyramidal number is $1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$. This can be visualized as the square pyramid shown below.



Construct three of these pyramids and try to form them into a brick. (It won't quite work; there will be a triangle, or staircase, left over.) Find the volume of the brick from the length of each side, and add it to the volume of the staircase. Divide by 3 (the number of pyramids), and see if you get 30. From this case of $n = 4$, guess at a general formula for the *n*th square pyramidal number. Guess at the length of each side of the brick in terms of *n*¸ and get its volume. And you have the generalization for the area of the staircase sticking out one side of the brick—the triangular number $n \times (n + 1) / 2$. Add these together, and divide by the number of pyramids.

5. Add the first *n* odd numbers together. (For $n = 4$ you get $1 + 3 + 5 + 7 = 16$.) Try this for several values of *n*, and see if you can guess a general formula for the sum of the first *n* odd numbers.

6. Here's a visual proof that $3 \times 4 = 4 \times 3$. (Rotation doesn't change anything.)



Can you give a visual proof that $1 + 3 = 3 + 1$?