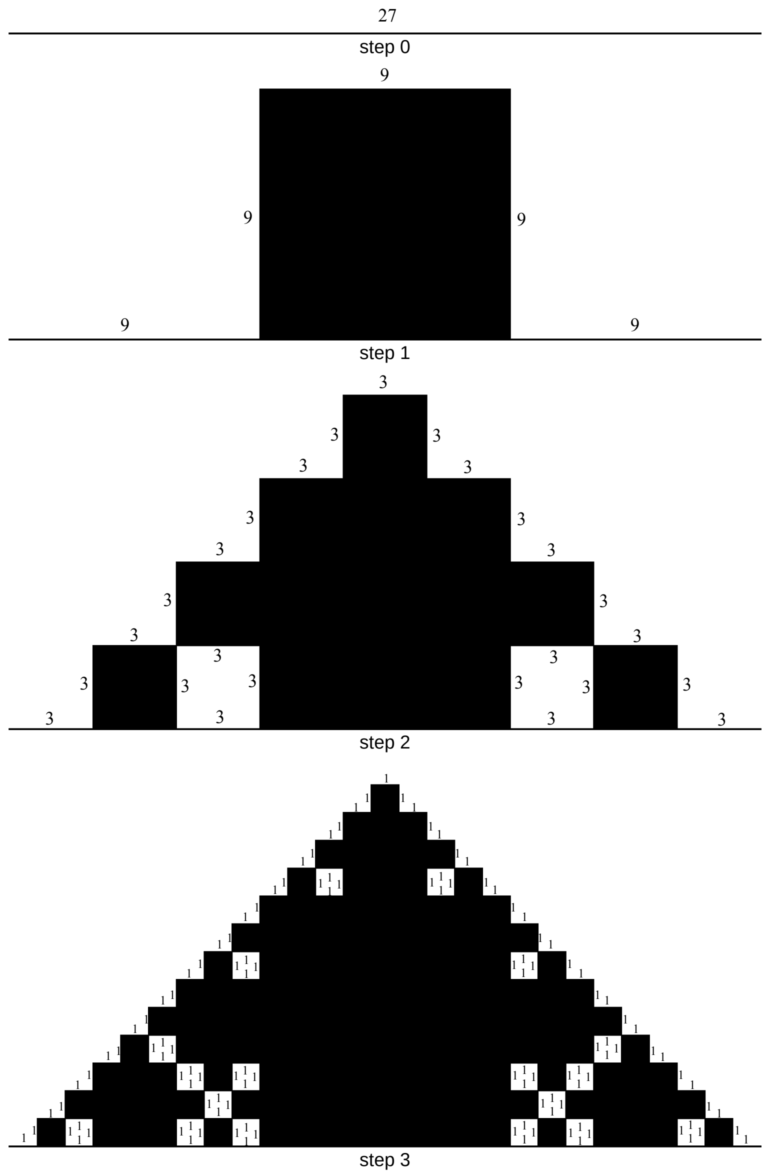
Lesson 2 – Answers
Fractals and Tiling

1.
Let the length of the line in Step 0 be **27**

In Step 1 a square with a side of 9 is placed on the middle third of the line. So the three lengths of 9 in Step 0 have been replaced by 5 lengths of 9 in Step 1. The perimeter in Step 1 is therefore (5/3) the length of the perimeter of 27 in Step 0.
(5/3)×27 = 5×9 = **45**.

In Step 2 a square with a side of 3 is placed on the middle of each of the five lines in Step 2. So each length of 9 in Step 1 has been replaced by 5 lengths of 3 in Step 2. The perimeter in Step 2 is therefore (5/3) the length of the perimeter of 45 in Step 1.
(5/3)×45 = 5×15 = 25×3 = **75**.

In Step 3 a square with a side of 1 is placed on the middle of each of the 25 lines in Step 2. So each length of 3 in Step 2 has been replaced by 5 lengths of 1 in Step 3. The perimeter in Step 3 is therefore (5/3) the length of the perimeter of 75 in Step 1.
(5/3)×75 = 5×25 = 125×1 = **125**.



So in each step the perimeter grows by a factor of (is multiplied by) 5/3. Therefore the perimeter grows arbitrarily large (becomes **infinite**).

Let the area in Step 1 be ***A***.

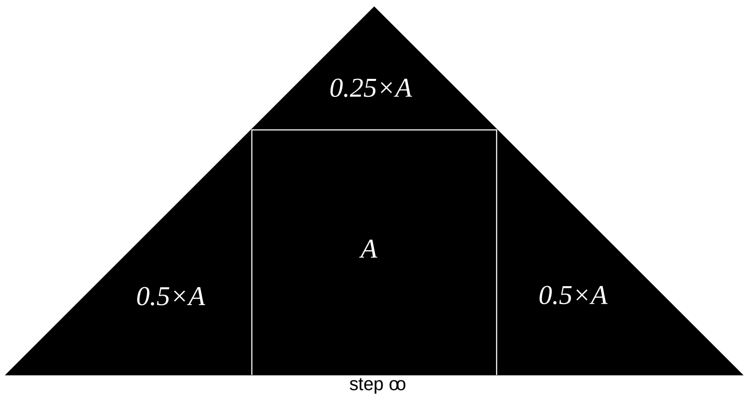
The five squares added in Step 2 are 1/3 the size of the square in Step 1, and they therefore have 1/9 the area: (1/9)×*A*. Therefore the new area in Step 2 is 5×(1/9)*A*, and the total area in Step 2 is *A* + 5×(1/9)*A* = ***A* + (5/9)*A***.

Each of the five new (small) squares in Step 2 gets five new (smaller) squares in Step 3. The area of each new square here is 1/9 the area of the new squares in Step 2: (1/9)×(1/9)×*A*. So the new area in Step 3 is 5×5×(1/9) ×(1/9) = (5/9) ×(5/9) = $(5/9)^2$, and the total area in Step 3 is **$A + (5/9)A + (5/9)^2 A$**.

In a similar way, the total area in Step 4 is **$A + (5/9)A + (5/9)^2 A + (5/9)^3 A$**, and the total area at Step ∞ is **$A + (5/9)A + (5/9)^2 A + (5/9)^3 A + \ldots$** But we have seen that this infinite sum can be expressed as

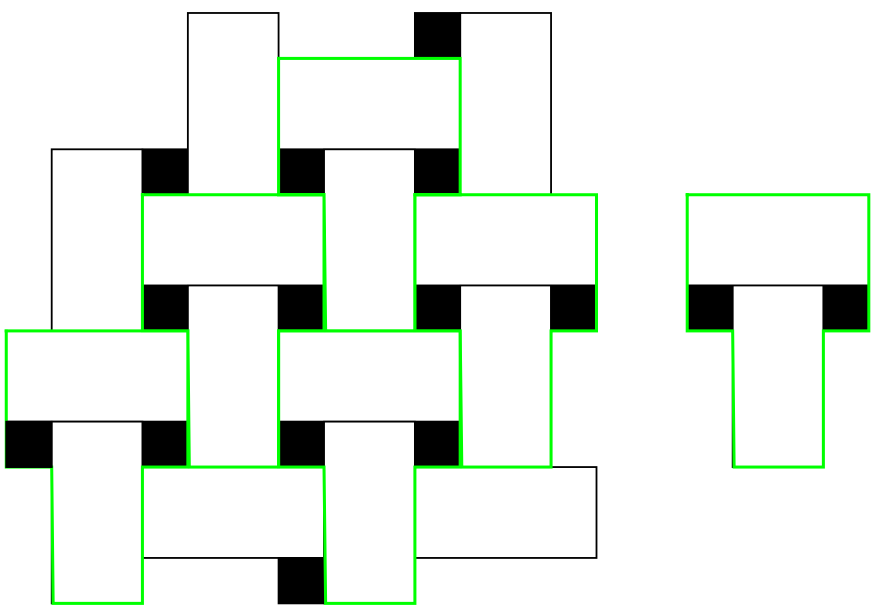
$$A + (5/9)A + (5/9)^2 A + (5/9)^3 A + \ldots = \frac{A}{1-(5/9)} = \frac{A}{(4/9)} = (9/4) \times A = 2.25 \times A$$

This agrees with what we see in Step ∞; the shape has become a triangle with area 2.25×*A*.



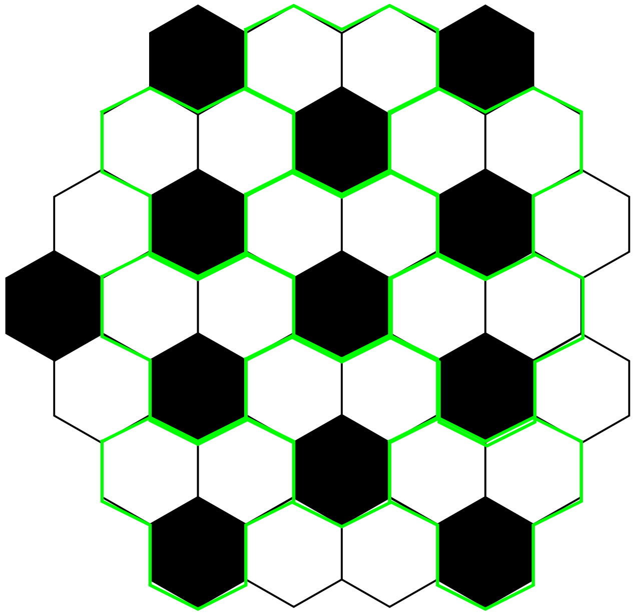
But this means that there are absolutely no square holes left; the holes are completely gone! And yet the perimeter is infinite, which means there are an infinite number of holes with zero size, and the sum of the distance around them is infinite. So infinity is a very strange thing to think about and try to understand.

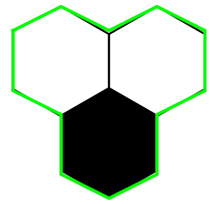
2.



Every two white tiles that form a T can be associated with the two black tiles under the horizontal part of the T. That is, we can say that the pattern on the left is made up of many of the small pattern on the right. So the number of white tiles is equal to the number of black tiles.

3.

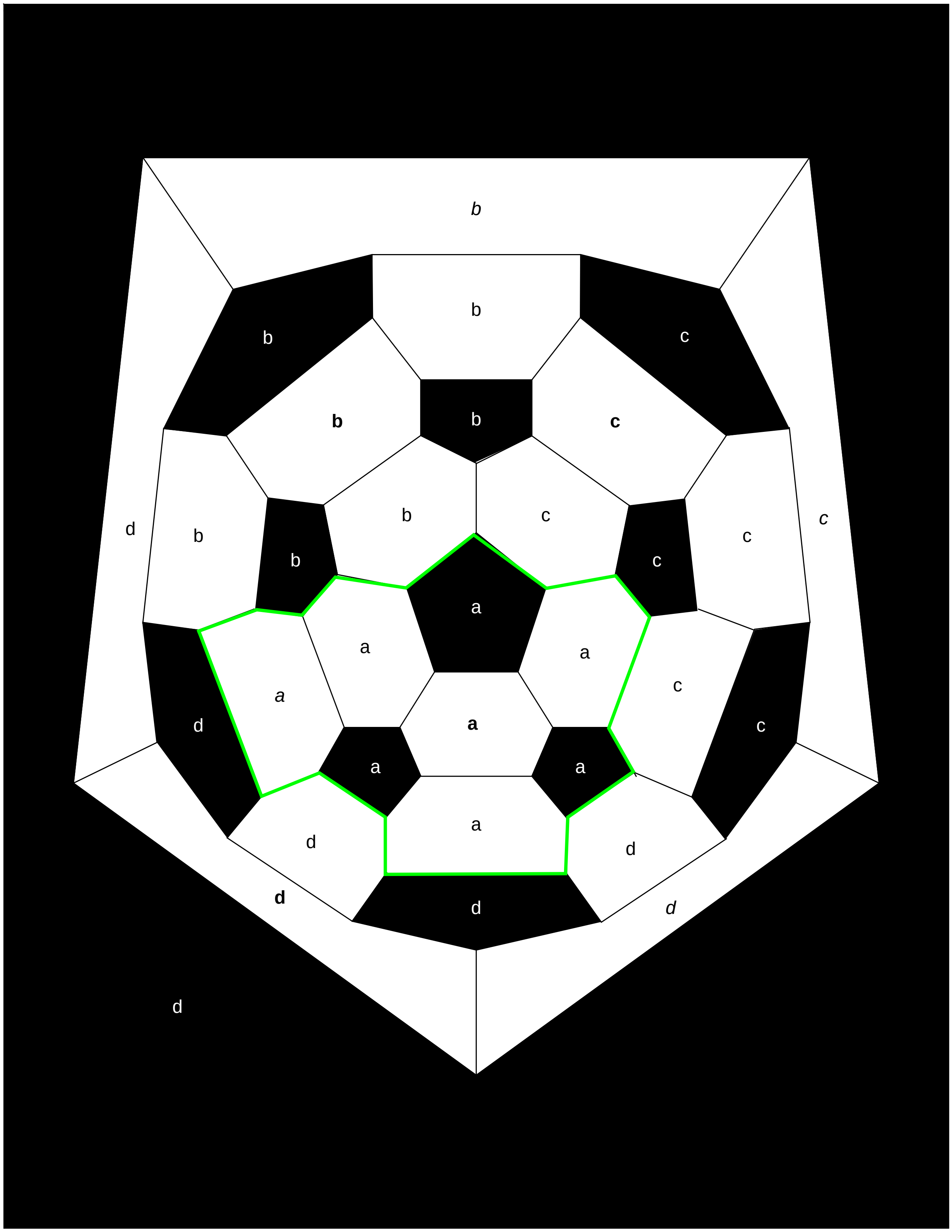




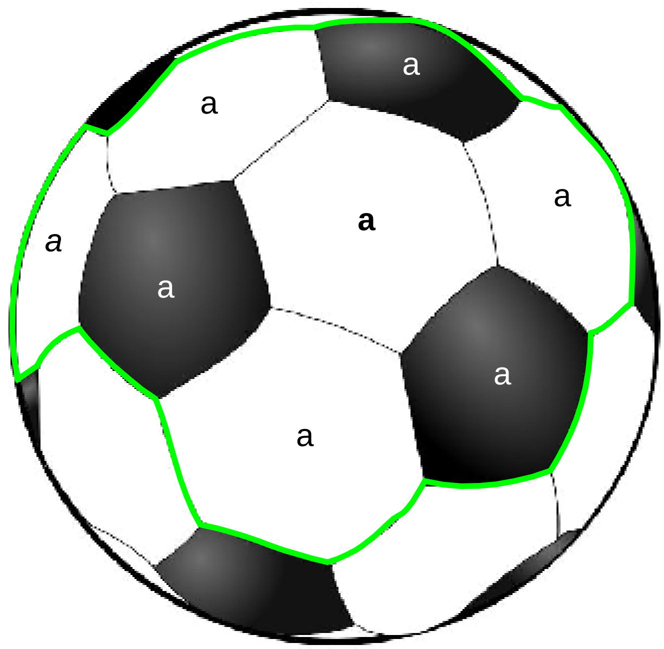
Every black tile can be associated with the two white tiles above it. That is, we can say that the pattern on the left is made up of many of the small pattern on the right. So the number of white tiles is twice the number of black tiles.

4.

There are 12 black pentagons and 20 white hexagons on the surface of a soccer ball. So we should be able to associate 3 black tiles with 5 white tiles five times in the same way. To be able to see all 32 tiles at the same time, we can tear a hole in the center of one of the black tiles and stretch the hole so it becomes the perimeter of a flat soccer ball, as shown below.



The green line encloses 3 black tiles and 5 white tiles labeled a. Three other similar groups are labeled b, c, and d. That is, we can say that a soccer ball is made up of four identical pieces like that bordered in green below. So there are 5 white tiles for every 3 black tiles.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