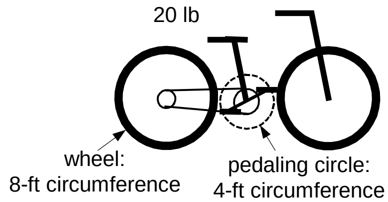
Chapter 9
Power and Optimum Performance

Example 1

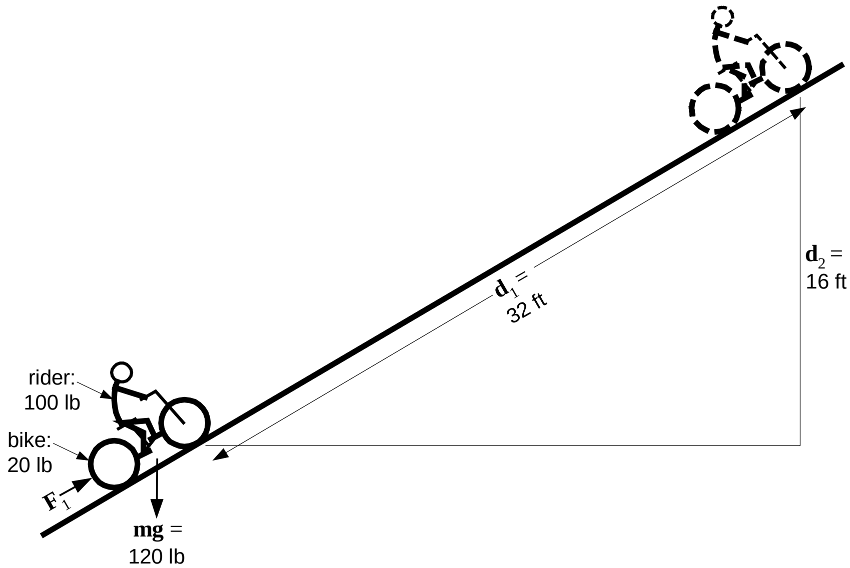


A bike weighs 20 lb, the circumference of its wheels is 8 ft, and the circumference of the pedaling circle is 4 ft. We define the gear ratio **R** as the number of times $\mathbf{N}_w$ the wheels turn divided by the number of times $\mathbf{N}_p$ the pedals go around.

$$\mathbf{R} = \mathbf{N}_w/\mathbf{N}_p$$

The ratios available are: 0.25, 0.35. 0.5, 0.7, 1.0, 1.4, 2.0, 2.8, 4.0, 5.6

A boy weighing 100 lb pedals up a 50% incline (rising 16 ft as he goes 32 ft). He uses gear ratio **R** = 1.0. How much work **W** does he do? How much force $\mathbf{F}_1$ is needed to push the bike up the incline? How much force $\mathbf{F}_3$ is needed on the pedals?



The work done equals the potential energy he gains:

$$\mathbf{W} = \mathbf{E}_p = \mathbf{mg} \cdot \mathbf{d}_2$$

where **mg** = 120 lb is the combined weight of the bike and rider, and $\mathbf{d}_2$ = 16 ft is the altitude he gains. Therefore **W** = (120 lb)·(16 ft) = 1920 ft-lb.

But the work **W** is also the force $\mathbf{F}_1$ in the direction of the incline times the distance $\mathbf{d}_1$ = 32 ft up the incline:

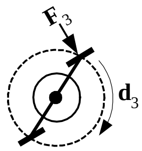
$$\mathbf{W} = \mathbf{F}_1 \cdot \mathbf{d}_1 = \mathbf{F}_1 \cdot (32\text{ ft})$$

so

$$\mathbf{F}_1 = (1920\text{ ft-lb})/(32\text{ ft}) = 60\text{ lb}.$$

To find the force $\mathbf{F}_3$ necessary on the pedal, we have to find the total distance $\mathbf{d}_3$ the pedal goes while the bike goes 32 ft. Since the wheel has a circumference of 8 ft, the wheel turns $\mathbf{N}_w = 32/8 = 4$ times. Since the rider has chosen the gear ratio $\mathbf{R} = 1.0$, the pedals go around $\mathbf{N}_p = \mathbf{N}_w / \mathbf{R} = 4/1.0 = 4$ times also. Since the pedaling circle circumference is 4 ft, the total distance the pedal goes is

$$\mathbf{d}_3 = \mathbf{N}_p \cdot 4 \text{ ft} = 4 \cdot 4 \text{ ft} = 16 \text{ ft}.$$



The work done by pedaling is $\mathbf{F}_3 \cdot \mathbf{d}_3$, where $\mathbf{F}_3$ is the force on the pedal. (We assume the force is constant, even though it shifts from pedal to pedal and is less when the pedals are at the top and bottom of the stroke.) It's still the same work $\mathbf{W}$ = 1920 ft-lb that's being done, so

$$\mathbf{F}_3 \cdot \mathbf{d}_3 = \mathbf{W} = 1920 \text{ ft-lb}$$

$$\mathbf{F}_3 = (1920 \text{ ft-lb})/\mathbf{d}_3 = (1920 \text{ ft-lb})/(16 \text{ ft}) = 120 \text{ lb}.$$

But the boy weighs only 100 lb, so even if he stands with his full weight on the pedal, he's not going to be able to move the bike uphill.

Example 2

The boy shifts to a lower gear ratio $\mathbf{R} = 0.5$ to reduce the force needed on the pedal. Then

$$\mathbf{N}_p = \mathbf{N}_w / \mathbf{R} = 4/0.5 = 8$$

$$\mathbf{d}_3 = \mathbf{N}_p \cdot 4 \text{ ft} = 8 \cdot 4 \text{ ft} = 32 \text{ ft}$$

$$\mathbf{F}_3 = (1920 \text{ ft-lb})/\mathbf{d}_3 = (1920 \text{ ft-lb})/(32 \text{ ft}) = 60 \text{ lb}.$$

This force on the pedal is no problem since it's less than the boy's weight of 100 lb.

What is the power $\mathbf{P}$ the boy is putting out? Power is defined as the work done per second; that is, power is the work divided by the time $\mathbf{T}$ taken to do the work:

$$\mathbf{P} = \mathbf{W}/\mathbf{T}.$$

How long does it take to go the $\mathbf{d}_1$ = 32 ft? Let's say the fastest the boy can turn the pedals (while applying $\mathbf{F}_3$ = 60 lb) is $\mathbf{r}$ = 1.0 revolution/second. Since the number of revolutions is $\mathbf{N}_p = 8$, the time is

$$\mathbf{T} = \mathbf{N}_p/\mathbf{r} = (8 \text{ rev})/(1.0 \text{ rev/s}) = 8 \text{ seconds},$$

$$\mathbf{P} = \mathbf{W}/\mathbf{T} = (1920 \text{ ft-lb})/(8 \text{ sec}) = 240 \text{ ft-lb/s}.$$

Now, one horsepower is 550 ft-lb/s, so the boy is putting out 240/550 = 0.436 hp of power. Not a horse, but not bad!

Example 3

The boy figures he can go the 32 ft in less time if he goes to a slightly higher gear. So he shifts to **R** = 0.7. Then

$$\mathbf{N}_p = \mathbf{N}_w / \mathbf{R} = 4/0.7 = 5.6$$

$$\mathbf{d}_3 = \mathbf{N}_p \cdot 4 \text{ ft} = 5.6 \cdot 4 \text{ ft} = 22.4 \text{ ft}$$

$$\mathbf{F}_3 = (1920 \text{ ft-lb})/\mathbf{d}_3 = (1920 \text{ ft-lb})/(22.4 \text{ ft}) = 85.7 \text{ lb}.$$

But this force is so close to the maximum he can give that it takes two seconds per revolution of the pedals. That is, the turning rate is **r** = 0.5 rev/sec. How long **T** does it take to go the $\mathbf{d}_1$ = 32 ft now?

$$\mathbf{T} = \mathbf{N}_p/\mathbf{r} = (5.6 \text{ rev})/(0.5 \text{ rev/s}) = 11.2 \text{ seconds}.$$

Now his power is

$$\mathbf{P} = \mathbf{W}/\mathbf{T} = (1920 \text{ ft-lb})/(11.2 \text{ sec}) = 171 \text{ ft-lb/s}.$$

So this is actually less power that he could deliver with **R** = 0.5.

Example 4

Maybe shifting to a lower gear **R** = 0.25 will let him put out more power. Then

$$\mathbf{N}_p = \mathbf{N}_w / \mathbf{R} = 4/0.25 = 16$$

$$\mathbf{d}_3 = \mathbf{N}_p \cdot 4 \text{ ft} = 16 \cdot 4 \text{ ft} = 64 \text{ ft}$$

$$\mathbf{F}_3 = (1920 \text{ ft-lb})/\mathbf{d}_3 = (1920 \text{ ft-lb})/(64 \text{ ft}) = 30 \text{ lb}.$$

This force is easy enough that the boy can pedal nearly as fast as possible—**r** = 1.7 rev/sec. How long **T** does it take to go the $\mathbf{d}_1$ = 32 ft now?

$$\mathbf{T} = \mathbf{N}_p/\mathbf{r} = (16 \text{ rev})/(1.7 \text{ rev/s}) = 9.4 \text{ seconds}.$$

Now his power is

$$\mathbf{P} = \mathbf{W}/\mathbf{T} = (1920 \text{ ft-lb})/(9.4 \text{ sec}) = 204 \text{ ft-lb/s}.$$

So this is still less power that he could deliver with **R** = 0.5.

Example 5

One more try: **R** = 0.37. Then

$$\mathbf{N}_p = \mathbf{N}_w / \mathbf{R} = 4/0.35 = 11.4$$

$$\mathbf{d}_3 = \mathbf{N}_p \cdot 4 \text{ ft} = 11.4 \cdot 4 \text{ ft} = 45.6 \text{ ft}$$

$$\mathbf{F}_3 = (1920 \text{ ft-lb})/\mathbf{d}_3 = (1920 \text{ ft-lb})/(45.6 \text{ ft}) = 42 \text{ lb}.$$

This force is still easy enough that the boy can pedal nearly as fast as possible—**r** = 1.4 rev/sec. How long **T** does it take to go the $\mathbf{d}_1$ = 32 ft now?

$$\mathbf{T} = \mathbf{N}_p/\mathbf{r} = (11.4 \text{ rev})/(1.4 \text{ rev/s}) = 8.14 \text{ seconds}.$$

Now his power is

$$\mathbf{P} = \mathbf{W}/\mathbf{T} = (1920 \text{ ft-lb})/(8.14 \text{ sec}) = 236 \text{ ft-lb/s}.$$

This is almost the same power of 240 ft-lb/s that he could deliver with **R** = 0.5.

Let's make a table of the results:

R (ratio)	0.25	0.35	0.5	0.7	1.0
r (rev/sec)	1.5	1.4	1.0	0.5	0
F (lb)	30	42	60	85.7	100
T (sec)	9.4	8.14	8.0	11.2	∞
P (ft-lb/s)	204	236	240	171	0

We see an optimum power at about **r** = 1.2 rev/sec and **F** = 50 lb (half the rider's weight). These numbers will depend, of course, on the rider.

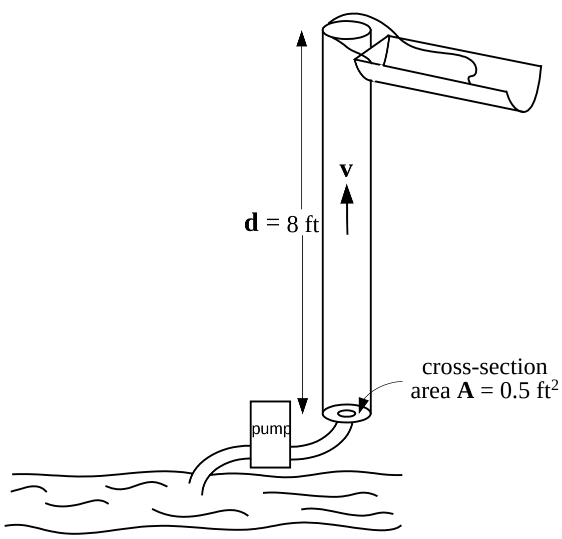
The definition of power is the work per second: **P** = **W**/**T**. But work is force through a distance: **W** = **F**·**d**. Therefore

$$\mathbf{P} = \mathbf{W}/\mathbf{T} = \mathbf{F}\cdot\mathbf{d}/\mathbf{T} = \mathbf{F}\cdot\mathbf{v},$$

where the force **F** and the velocity **v** are in the same direction. For instance, in this example the bike went 32 ft in 8.14 seconds, so $\mathbf{v}_1 = 32/8.14 = 3.93$ ft/s. The force $\mathbf{F}_1$ (on the bike, not on the pedal) is 60 lb. So the power is $\mathbf{P} = \mathbf{F}_1\cdot\mathbf{v}_1 = (60 \text{ lb})\cdot(3.93 \text{ ft/s}) =$ 236 ft-lb/s, which is the same as we got from **P** = **W**/**T** = (1920 ft-lb)/(8.14 sec) = 236 ft-lb/s.

Example 6

Let's apply this equation **P** = **F**·**v** to the hydraulic system shown below to find the power



the pump must provide. The pump takes water from a stream and fills a pipe **d** = 8 ft tall with a cross-section area of **A** = 0.5 ft^2 (the diameter is 9.5 inches). When the pipe is filled, it has (8 ft)·(0.5 ft^2) = 4 ft^3 of water. Since a cubic foot of water weighs 62 pounds, the total weight of the column of water is (4 ft^3)·(62 lb/ft^3) = 248 lb.

As the pump continues, it lifts the 248-lb column with a velocity of **v** = 2 ft/s. Since the cross-section is **A** = 0.5 ft^2, the flow-rate with which water spills from the top is

$$\mathbf{f} = \mathbf{A}\cdot\mathbf{v} = (0.5\ ft^2)\cdot(2\ ft/s) = 1\ ft^3/s.$$

(This water goes into a trough for irrigation.) To find the power provided by the pump, we need to know the force **F** required to lift the column of water as well as the column's velocity **v**. The force is just the weight of the column, so **F** = 248 lb, and we know **v** = 2 ft/s. Therefore

$$\mathbf{P} = \mathbf{F}\cdot\mathbf{v} = (248\ lb)\cdot(2\ ft/s) = 496\ ft\text{-}lb/s.$$

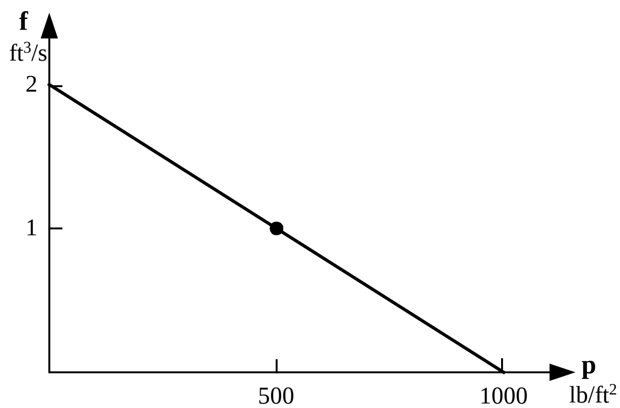
The pump knows nothing of the pipe's dimensions; all it knows is the flow-rate **f** = 1 ft^3/s it is providing and the pressure it is encountering. The pressure **p** at the bottom of the pipe is the total weight **F** = 248 lb divided by the area **A** at the bottom of the pipe:

$$\mathbf{p} = \mathbf{F}/\mathbf{A} = (248\ lb)/(0.5\ ft^2) = 496\ lb/ft^2.$$

We can show that the power **P** can also be calculated from the flow-rate **f** and the pressure **p**:

$$\mathbf{P} = \mathbf{p}\cdot\mathbf{f} = (496\ lb/ft^2)\cdot(1\ ft^3/s) = 496\ ft\text{-}lb/s.$$

The flow **f** that a pump can provide depends on the pressure it encounters. The plot below shows a typical relationship; as the pressure **p** increases, the flow decreases until it goes to zero at some maximum pressure (1000 lb/ft^2 here).



Our pump is operating at the point shown by the dot.

The electrical power provided to the pump must be about the same as the mechanical power **P** = 496 ft-lb/s. Since electrical power is in watts, we need to convert using 1.356 watts per ft-lb/s.

$$\mathbf{P} = (496\ ft\text{-}lb/s)\cdot(1.356\ \text{watts per ft-lb/s}) = 672\ \text{watts}.$$

In the next lesson we will see that electrical power is given by

$$\mathbf{P} = (\text{voltage})\cdot(\text{current}),$$

where the power is in watts, the voltage is in volts, and the current is in amperes. Suppose the voltage supplied to the pump is 120 volts. Then we can find the current:

$$\text{current} = \mathbf{P}/\text{voltage} = (672\ \text{watts})/(120\ \text{volts}) = 5.6\ \text{amperes}.$$

Problem

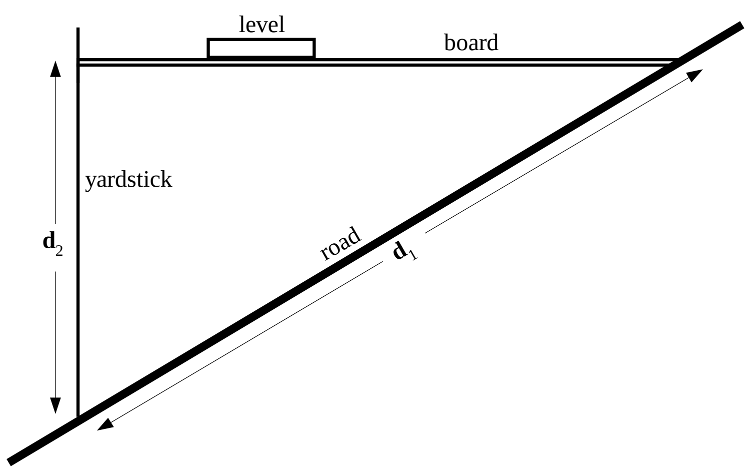
The 8-ft pipe in Example 6 is reduced to a cross-section of $\mathbf{A} = 0.1 \text{ ft}^2$ and the velocity is increased to $\mathbf{v} = 10$ ft/s. Find the new force **F**, the new pressure **p**, the new flow-rate **f**, and the new power **P** (calculated by $\mathbf{P} = \mathbf{F} \cdot \mathbf{v}$ and by $\mathbf{P} = \mathbf{p} \cdot \mathbf{f}$). Does the pump know anything about this change? Evidently water pressure at the bottom of the pipe depends on just one dimension (such as **d**, **A**, **v**, etc.); what is it?

Experiment

Get to know you and your bike. Find

1. Your weight
2. The bike's weight
3. The wheel circumference
4. The pedaling circle circumference
5. The gear ratios **R** (either use a chalk mark on the tire, or count the teeth on the gears)

Find the grade **G** of a road incline. Get a long straight board and make it level, with one end on the road. Measure the height $\mathbf{d}_2$ of the other end above the road. Measure the length $\mathbf{d}_1$ on the road from the yardstick to the board. The grade is $\mathbf{G} = \mathbf{d}_2/\mathbf{d}_1$.



Mark out a longer $\mathbf{d}_1$ on the road (about 30 ft), and calculate the new $\mathbf{d}_2$ from $\mathbf{G} = \mathbf{d}_2/\mathbf{d}_1$. What work will you have to do to gain the elevation $\mathbf{d}_2$?

Using different gears, ride your bike as fast as you can over the length $\mathbf{d}_1$ you've marked. (Rest up a couple minutes between rides.) Each time, have someone with a stopwatch measure the time **T**. For each run calculate **r**, **F**, and **P**. Make a table of the values **R**, **r**, **F**, **T**, and **P** as in Example 5. What were **r** and **F** for your highest **P**? What was the highest **R** for which you could go up the incline? Was the corresponding **F** just less than your weight?