Chapter 8
Pendulums

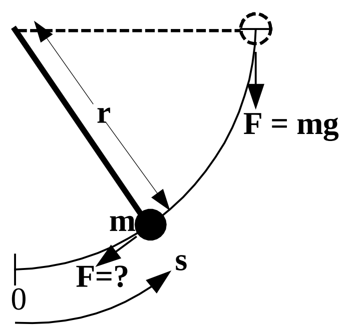
A pendulum is a form of mechanical oscillator, such as we looked at in Lesson 7. IN this case the restorative force (that pushes the pendulum back toward the center) is provided by gravity rather than by a spring. If the restorative force is proportional to the displacement (distance from the center), it will be a harmonic oscillator with sine wave motion and with a constant period—independent of the size of its swing. We want to find out whether the restorative force is, in fact, proportional to displacement, and we want to know how the period depends on the mass and on the length of the pendulum.

Design Problem

The pendulum of a grandfather clock is to have a period of

$$\mathbf{T} = 2 \text{ seconds.}$$

What is the length **r** of the pendulum? What should the mass **m** be? How far **s** can it swing and still keep good time?



We're hoping that a pendulum oscillator is like a spring oscillator, that is, it's a harmonic oscillator. If it is, then it period **T** is determined by

$$\mathbf{T}^2 = 6.28^2 \cdot \mathbf{m} \cdot \mathbf{k}_s$$

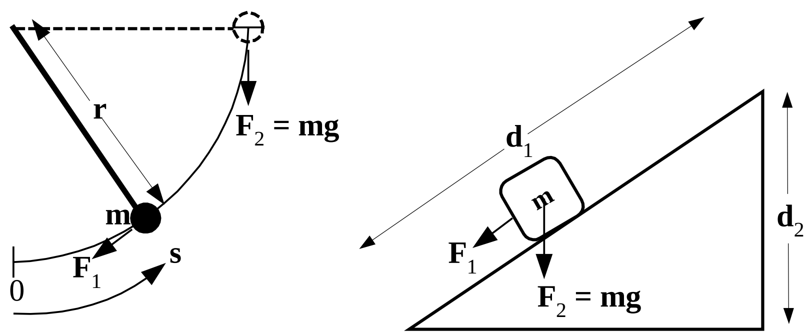
where $\mathbf{k}_s$ is the spring constant (see Lesson 7). There's no spring here, but gravity can act like a spring if the displacement **s** is proportional to the restorative force **F** due to gravity:

$$\mathbf{s} = \mathbf{k}_s \cdot \mathbf{F}$$

We know that **F** equals the weight **mg** of the mass when the pendulum is horizontal. Then the mass is swinging straight down—in the direction of gravity. But when the pendulum is swinging sideways (with only a small downward component), we sense that the force **F** will be less. How do we calculate exactly how much less?

The pendulum is constraining the mass to move at an angle to gravity, not straight up and down. But this is like Problem 3 in Lesson 5 where we were pushing a block of ice up an inclined ramp. The ramp constrained the mass to move at an angle to gravity, not straight up and down (see the figure below). We found that the work done pushing the mass up a height $\mathbf{d}_2$ had to be the same whether we went on an angle through a distance $\mathbf{d}_1$ or straight up through a distance $\mathbf{d}_2$. But work is $\mathbf{W} = \mathbf{F} \cdot \mathbf{d}$, where **F** and **d** are in the same direction. That is,

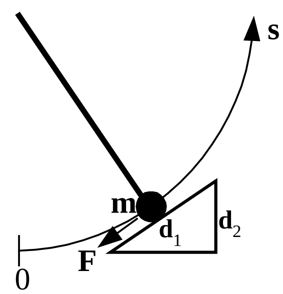
$$\mathbf{W} = \mathbf{F}_1 \cdot \mathbf{d}_1 = \mathbf{F}_2 \cdot \mathbf{d}_2$$



The longer the distance, the smaller the force required. So we can solve for $\mathbf{F}_1$:

$$\mathbf{F}_1 = \mathbf{F}_2 \cdot \mathbf{d}_2/\mathbf{d}_1 = \mathbf{mg} \cdot \mathbf{d}_2/\mathbf{d}_1$$

We can pretend that the mass on the end of our pendulum goes at an angle because of a small ramp that keeps changing its angle, as in the figure below.

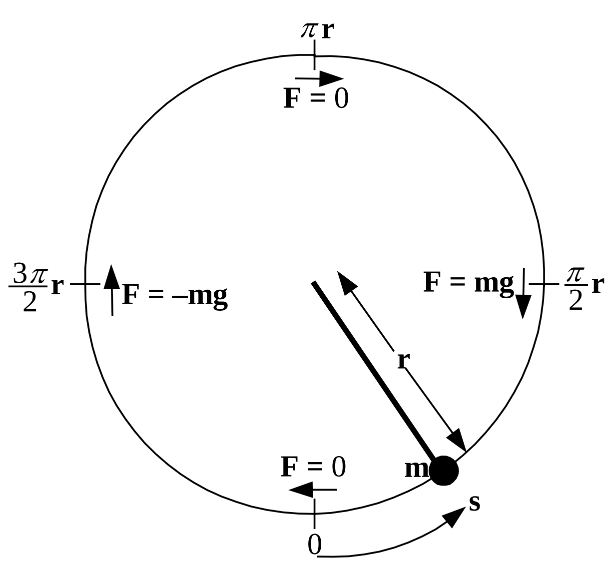


When the pendulum is vertical, then $\mathbf{d}_2$ goes to zero, and $\mathbf{F} = 0$. When the pendulum is horizontal, then $\mathbf{d}_2 = \mathbf{d}_1$, and $\mathbf{F} = \mathbf{mg}$. In between these two positions, the ratio $\mathbf{d}_2/\mathbf{d}_1$ changes smoothly, and **F** changes smoothly.

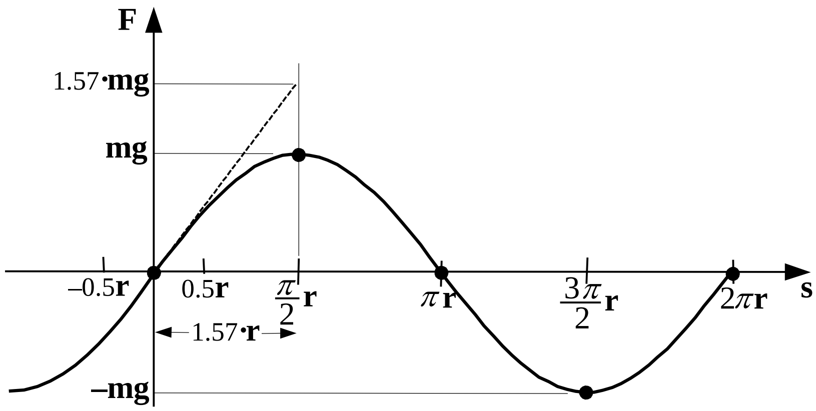
To find our "spring constant" $\mathbf{k}_s$ we need to know what **F** is for each distance **s** since

$$\mathbf{k}_s = \mathbf{s}/\mathbf{F}.$$

($\mathbf{k}_s$ is small when the swing **s** increases only a small amount for a given force **F**.) The figure below shows four values of **s** corresponding to positions where we know what **F** is.



When the pendulum is horizontal, then **s** = (π/2)**r**, which is ¼ the circumference of the circle, and **F** = **mg**. Although we usually don't swing the pendulum so far, at the top of the circle **s** = π·**r**, which is half the circumference, and the force clockwise (along the circle) is again zero (**F** = 0). If we swing even farther to **s** = (3π/2)**r**, then gravity is pulling against the restorative force **F** (clockwise), and **F** = **–mg**. When **s** = 2π·**r**, we are back to the same position as **s** = 0, and **F** = 0. The figure below plots these five positions and the corresponding clockwise forces. The value of **F** changes smoothly between the points.



As expected, the curve is a sine wave. Is the force **F** proportional to the swing distance **s**? No, it's proportional only if the plot is a straight line. But for a small region around **s** = 0 we can say that the curve approximates a straight line—between about –0.5**r** and 0.5**r**. So for the clock to keep good time, we don't let the pendulum swing too far. (Actually, 0.5**r** isn't all that small; it's about the position show for the pendulum in all the figures above.)

So we can get a "spring constant" for the pendulum in the region –0.5**r** < **s** < 0.5**r**. As we've done before, we extend the slope in that region until it reaches a point directly above the maximum of the sine wave. At that point the extension will have reached 57% higher than the maximum value **mg** of the sine wave, that is, its rise is 1.57·**mg**. The run during that rise is (π/2)**r**, or 1.57·**r**. The slope is the rise divided by the run:

$$\text{slope} = (1.57\cdot\mathbf{mg})/(1.57\cdot\mathbf{r}) = \mathbf{mg}/\mathbf{r}.$$

So in that region **F** = (slope)·**s** = (**mg**/**r**)·**s**. But the spring constant is $\mathbf{k}_s = \mathbf{s}/\mathbf{F}$. Therefore,

$$\mathbf{k}_s = \mathbf{r}/(\mathbf{mg}).$$

Now we can go back to our equation for the period **T**:

$$\mathbf{T}^2 = 6.28^2\cdot\mathbf{m}\cdot\mathbf{k}_s = 6.28^2\cdot\mathbf{m}\cdot\mathbf{r}/(\mathbf{mg}) = 6.28^2\cdot\mathbf{r}/\mathbf{g}.$$

Notice that the two **m**'s cancel out, so the period of a pendulum is independent of the mass we use (until the mass becomes small compared to the weight of the pendulum arm). We know **g** = 386 in/s^2, so we need to choose the pendulum length **r** so that **T** = 2 seconds. Solving the above equation for **r**, we get

$$\mathbf{r} = \mathbf{T}^2\mathbf{g}/(6.28^2) = 2^2(386)/(39.5) = 39.2 \text{ inches}.$$

This is the length of the pendulum in a grandfather clock.

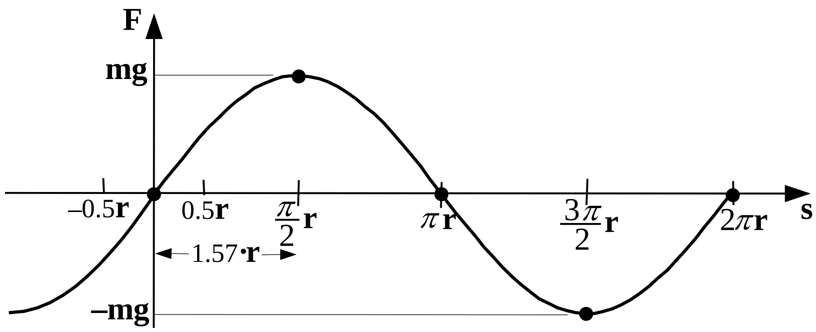
Problems

Problem 1

Find the length **r** of a pendulum so its period **T** is 1 second.

Problem 2

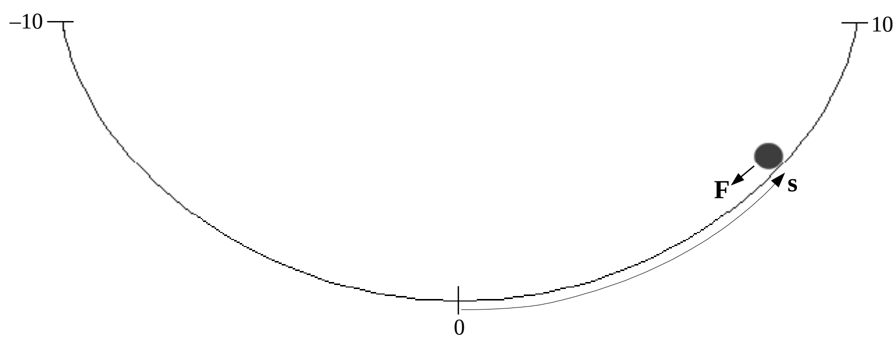
When a pendulum swings so far that it is horizontal, the effective "spring constant" at that instant is larger than the $\mathbf{k}_s = \mathbf{r}/(\mathbf{mg})$ we found for small swings. Find the "spring constant" when $\mathbf{s} = (\pi/2)\mathbf{r}$, that is, when the pendulum is horizontal.



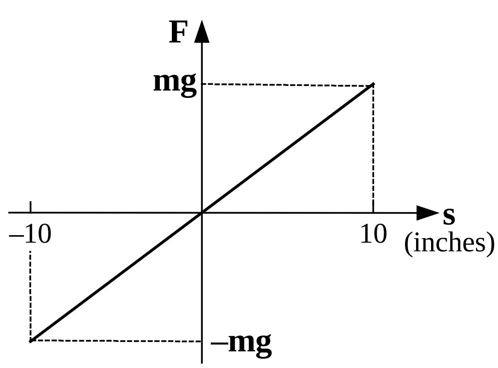
Remember that at every point of the swing, the spring constant is defined as $\mathbf{k}_s = \mathbf{s}/\mathbf{F}$. If the pendulum had that larger $\mathbf{k}_s$ for the whole swing, what would the expression for $\mathbf{T}^2$ be? (It will be larger than $\mathbf{T}^2 = 6.28^2 \cdot \mathbf{r}/\mathbf{g}$.)

Problem 3

We have a special bowl whose cross-section is a cycloid (see Wikipedia).



We put a steel ball in the bowl and let it roll back and forth. It turns out that, for this shape of bowl, the restorative force **F** is always proportional to the displacement **s** from the bottom of the bowl for any swing **s** up to 10 inches (see Figure below).



What is the "spring constant"? What is the period **T** of this oscillator? (**T** will actually be a little larger than your calculation because some of the force of gravity is "wasted" in rotating the ball rather than accelerating its center of gravity.) Note that **T** is absolutely independent of how far the ball rolls back and forth.

Experiments

Experiment 1

Construct a pendulum with a string and a wood ball so that its period is 1 second (see Problem 1). Measure its period for small swings and see how close to 1 second it is.

Experiment 2

Use your pendulum from Experiment 1 with a large swing, releasing the ball when the string is horizontal. Measure **T**. Is it larger than the **T** you measured in Experiment 1? Is it less than the results of Problem 2 using your value of **r** in Experiment 1?