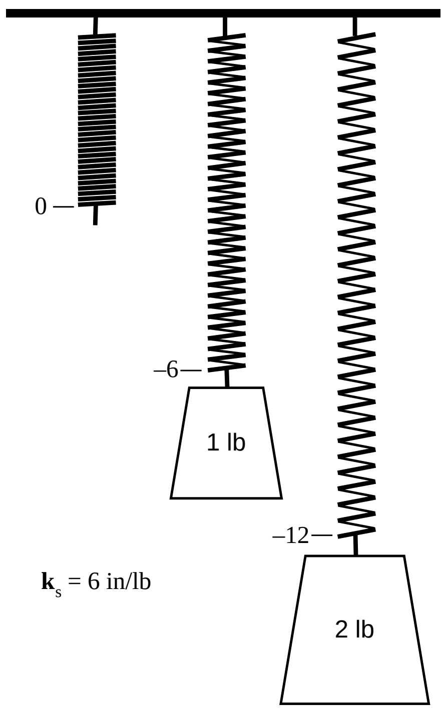
Chapter 7
Springs and Mechanical Oscillators

Springs

When you pull on a spring, it pulls back with a force $\mathbf{F}_s$ proportional to the distance **d** you pull it.



For the spring shown, if you pull it with 1 lb of force, the distance **d** is 6 inches. If you pull it with 2 lb of force, the distance is 12 inches. The relationship is

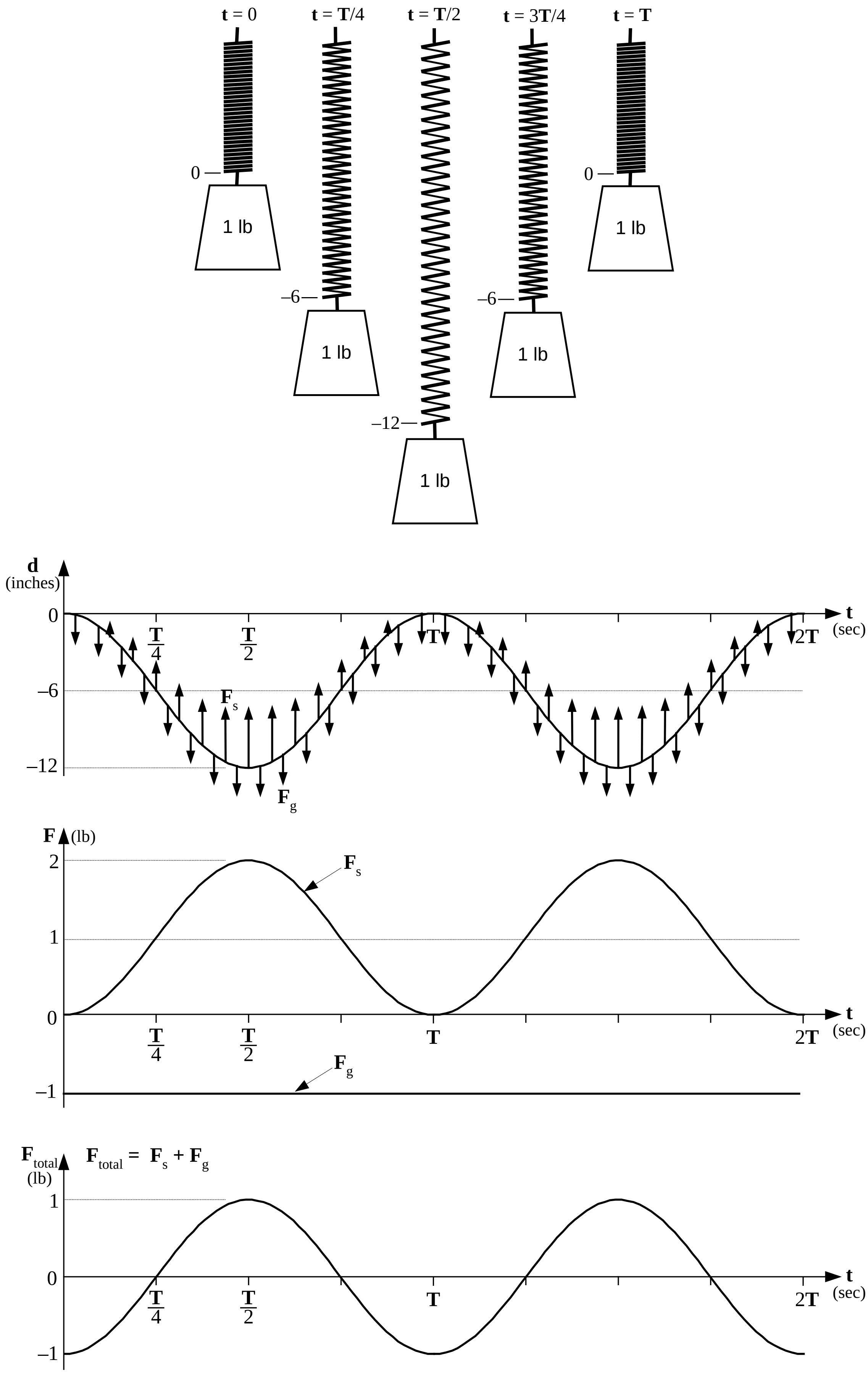
$$\mathbf{d} = \mathbf{k}_s\mathbf{F}_s$$

where $\mathbf{k}_s$ is the *spring constant*. In our case, $\mathbf{k}_s$ = 6 in/lb; that is, it stretches 6 inches for every pound of force you apply.

Oscillator

We attach a 1-lb weight to the spring and let it go. At **t** = 0 it has just been let go (see figure next page). The acceleration of gravity hasn't had a change to move the weight yet, and **d** = 0. After some time **T**/4 the spring has stretched to **d** = 6 inches, after a time **T**/2 the spring has stretched to **d** = 12 inches, and after a time **T** the spring has relaxed back to **d** = 0. The weight will continue to bounce this way, repeating a *cycle* every *period* **T**. The shape of the curve with time is a sine wave (see plot for **d** next page).

As the spring stretches (**d** goes negative), the force $\mathbf{F}_s$ of the spring upward increases. We illustrate this by upward arrows that get longer as the spring gets longer (see plot for **d** next page). At the same time, gravity is pulling with a constant force $\mathbf{F}_g$ = –1 lb. We illustrate this by downward arrows that are a constant length. When we combine these forces we get $\mathbf{F}_{total} = \mathbf{F}_s + \mathbf{F}_g$, which is –1 lb at the top of the bounce (**d** = 0) and +1 lb at the bottom (**d** = –12).

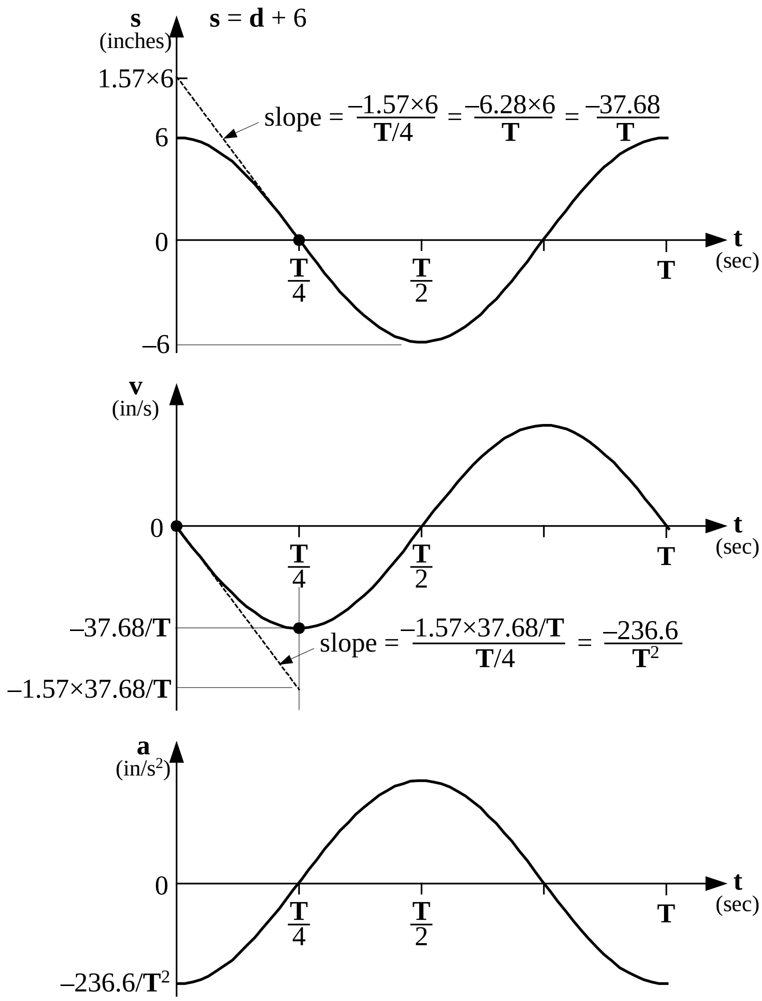


t = 0
t = T/4
t = T/2
t = 3T/4
t = T
0
–6
–12
–6
0
1 lb
1 lb
1 lb
1 lb
1 lb
d
(inches)
0
–6
–12
T/4
T/2
T
2T
t
(sec)
Fs
Fg
F (lb)
2
1
0
–1
T/4
T/2
T
2T
t
(sec)
Fs
Fg
Ftotal
(lb)
Ftotal = Fs + Fg
1
0
–1
T/4
T/2
T
2T
t
(sec)

We don't know what the period **T** is yet, but we can calculate it from the relation **F** = **ma**. To find the acceleration **a**, we need to find some slopes of the **d** curve. First we shift our plot of distance so zero distance is defined as the middle of the swing. So we let

$$\mathbf{s} = \mathbf{d} + 6.$$

Then **s**= 0 at **t** = **T**/4, and **s** goes from 6 to –6.



The velocity **v** of the mass is the slope of the distance curve **s**. At time **t** = **T**/4 the slope is the steepest downward (most negative velocity). We can find the slope there by extending a straight line from that point until it hits the **s** axis at **t** = 0. Since we're dealing with a sine wave, our rule is that it will hit 57% higher than the maximum value of **s**, which is 6 inches. That is, it hits the **s** axis at 1.57×6 (see the plot for **s** above).

The slope of the line (and therefore the slope of the **s** curve at **t** = **T**/4) is the *rise* 1.57×6 divided by the *run* **T**/4. Since the slope is downward, it's a negative slope.

$$\text{slope} = \frac{-1.57\times 6}{\mathbf{T}/4} = \frac{-6.28\times 6}{\mathbf{T}} = \frac{-37.68}{\mathbf{T}}$$

This is the minimum (most negative velocity). It turns out the velocity **v** is also a sine wave. It has the same period **T**, and we know its *amplitude* $\mathbf{v}_{max}$ = 37.68/**T**, so we can plot it (see the figure above).

The acceleration **a** of the mass is the slope of the velocity curve **v**. At time **t** = 0 the slope is the steepest downward (most negative acceleration). We can find the slope there by extending a straight line from that point to the time $\mathbf{t} = \mathbf{T}/4$. Again, our rule is that it will reach 57% farther than the value of **v** there, which is $-37.68/\mathbf{T}$ in/s. That is, it reaches $1.57(-37.68/\mathbf{T})$ (see the plot for **v** above).

The slope of the line (and therefore the slope of the **v** curve at $\mathbf{t} = 0$) is the rise $-1.57\times 37.68/\mathbf{T}$ divided by the run $\mathbf{T}/4$. Since the slope is downward, it's a negative slope.

$$\text{slope} = \frac{-1.57\times 37.68/\mathbf{T}}{\mathbf{T}/4} = \frac{-236.6}{\mathbf{T}^2}$$

This is the minimum (most negative acceleration). It turns out the acceleration **a** is also a sine wave. It has the same period **T**, and we know its amplitude $\mathbf{a}_{max} = 236.6/\mathbf{T}^2$, so we can plot it (see the figure above).

So at time $\mathbf{t} = 0$ we know the force $\mathbf{F}_{total\,=}$ –1 lb, the acceleration $\mathbf{a} = -236.6/\mathbf{T}^2$ in/s^2, and the mass **m**:

$$\mathbf{m} = (1\text{ lb})/\mathbf{g} = (1\text{ lb})/(386\text{ in/s}^2).$$

So from $\mathbf{F}_{total} = \mathbf{ma}$ we can solve for **T**.

$$\mathbf{F}_{total} = [\mathbf{m}][\mathbf{a}]$$

$$-1\text{ lb} = [(1\text{ lb})/(386\text{ in/s}^2)][-236.6/\mathbf{T}^2\text{ in/s}^2] = (1\text{ lb})\,(-0.61/\mathbf{T}^2)$$

$$\mathbf{T}^2 = 0.61\text{ s}^2$$

$$\mathbf{T} = 0.78\text{ sec}$$

(Try multiplying 0.78 by itself, and show that the result is 0.61.) So the period of the oscillator is a little less than 1 second. If the mass were a little larger or the spring were a little stretchier (had a larger $\mathbf{k}_s$), then the period would be longer.

If we had done the calculation for our oscillator using the symbol $\mathbf{k}_s$ instead of the specific 6 in/lb, and using the symbol **m** instead of the specific (1 lb)/(386 in/s^2), we would get the equation

$$\mathbf{T}^2 = 6.28^2\cdot\mathbf{m}\cdot\mathbf{k}_s, \qquad \mathbf{T} = 6.28\cdot\sqrt{\mathbf{m}\cdot\mathbf{k}_s}$$

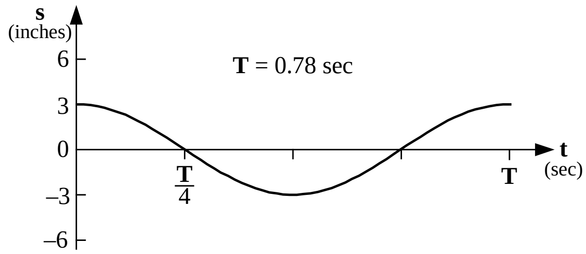
For example, if $\mathbf{m} = (1\text{ lb})/(386\text{ in/s}^2)$ and $\mathbf{k}_s = 6$ in/lb, then

$$\mathbf{T} = 6.28\cdot\sqrt{0.0155\text{ sec}^2}$$

$$= 6.28\cdot 0.125 = 0.78\text{ sec}$$

as before.

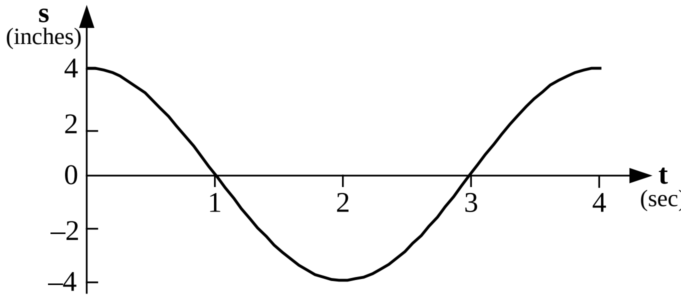
Notice that the amplitude of the oscillation doesn't affect **T**. If the mass bounced only 3 inches above and 3 inches below the center point, the period would still be $\mathbf{T} = 0.78$ sec.



This is true for all harmonic oscillators. We get harmonic oscillation if the restorative force is proportional to the displacement. That is, the force pulling the mass back toward the center is proportional to how far it is from the center. If this is true then the oscillation will be sinusoidal. We will see in the next lesson that a pendulum is almost a harmonic oscillator—until the amplitude gets too large.

Problems

Problem 1



An oscillator bounces 4 inches each side of center with a period **T** = 4 sec. Plot both the velocity and acceleration of the mass. What is its maximum acceleration?

If the mass weighs 1 lb, what is its mass **m**? What is the maximum force on the mass (corresponding to the maximum acceleration)?

From the maximum force and the maximum displacement **s**, find the spring constant $\mathbf{k}_s$.

See if the equation $\mathbf{T}^2 = 6.28^2 \cdot \mathbf{m} \cdot \mathbf{k}_s$ holds.

Problem 2

You have a spring with a spring constant of 6 in/lb, and you want the period to be exactly 1 second. What mass **m** must you choose? What is the weight of that mass?

Experiments

Experiment 1

Construct an oscillator with a mass and a spring. Find a weight of about 0.5 lb (check its weight on a kitchen scale). What is its mass **m**? Use that weight to find the spring constant $\mathbf{k}_s$ of your spring. Measure the period **T** of your oscillator. You can measure several periods for more accuracy (a little tug when the weight is at the bottom will keep the oscillations going). See if the equation $\mathbf{T}^2 = 6.28^2 \cdot \mathbf{m} \cdot \mathbf{k}_s$ holds.

Measure the period for two or three different amplitudes of oscillation. Does the period change?

Experiment 2

Repeat Experiment 1 with a larger weight.