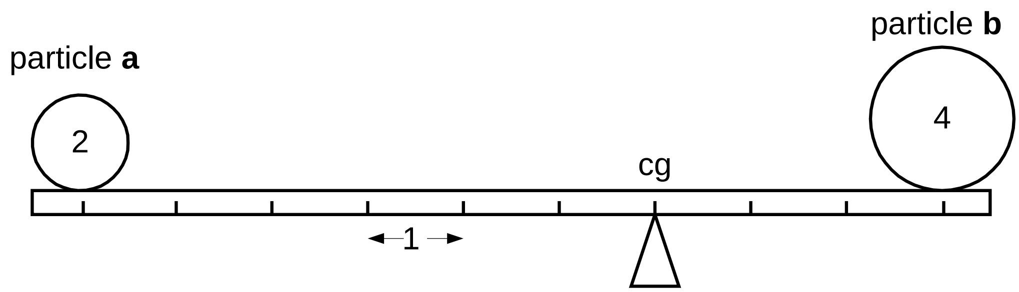
Chapter 4
Momentum and Center of Gravity

We’ve been treating a mass (a ball, for instance) as a single mass, while in reality it’s made up of many atoms that are moving (vibrating) with thermal energy. When the center of gravity of the atoms moves, we say the ball is moving and has kinetic energy. We can simplify the problem by ignoring the thermal energy inside the ball. Suppose we were given the positions and velocities of all the atoms in the ball. How would we find ball’s center of gravity and the velocity of that center of gravity?

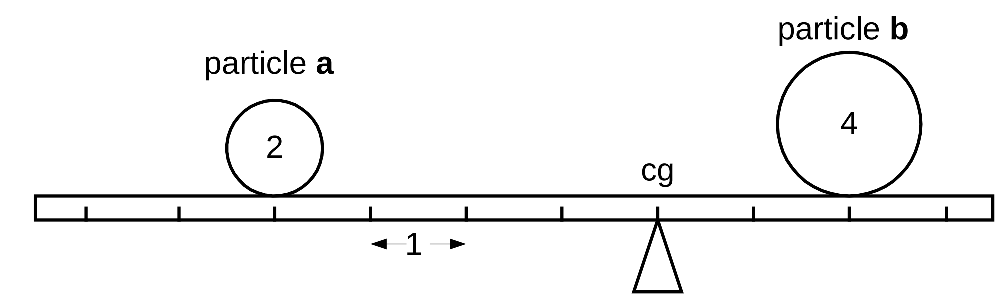
Example 1

Let’s look at a simple group of two particles 9 inches apart—particle **a** with a mass of 2 and particle **b** with a mass of 4—and they’re both at rest (not moving). Where is the center of gravity of the group, and how fast is it moving? We find the center of gravity by supporting the two

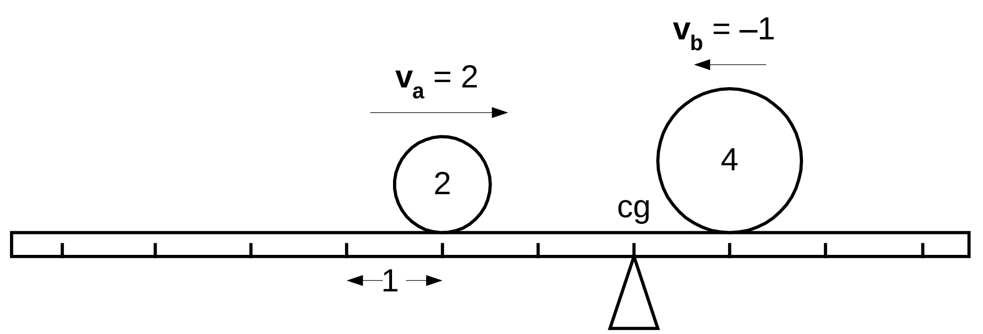


particles on a teeter-totter and find where it balances. The balance point (the center of gravity cg) is 3 inches from particle **b** and 6 inches from the particle **a** (twice as far because it’s half the mass). The center of gravity is not moving (has a velocity of 0).

Now particle **a** moves toward the center of gravity, going 2 inches in one second. If the center of gravity is to stay n the same place (keep a velocity of 0), particle **b** must move 1 inch closer to the center of gravity.



Particle **a** again moves toward the center of gravity by 2 inches in one second. If the center of



gravity is to stay n the same place (keep a velocity of 0), particle **b** must move 1 inch closer to the center of gravity. So the velocity $\mathbf{v_b}$ of particle **b** is 1 in/sec, and the velocity $\mathbf{v_a}$ of particle **a** is 2 in/sec—twice as much. Our feeling that cg doesn't move is based on an intuitive sense. Now we'll analyze the situation more carefully.

Let's find the energy and momentum of each particle and of the center of gravity. Remember that the relationships are

Energy: $\mathbf{E} = \frac{1}{2}\, \mathbf{m}\, \mathbf{v}^2$

Momentum: $\mathbf{P} = \mathbf{m}\, \mathbf{v}$

Make a table with the values to keep track of what we're doing:

Reference Frame 1:

$\mathbf{v_a}$	$\mathbf{v_a}^2$	$\mathbf{m_a}$	$\mathbf{E_a}$	$\mathbf{P_a}$
2	4	2	4	4

$\mathbf{v_b}$	$\mathbf{v_b}^2$	$\mathbf{m_b}$	$\mathbf{E_b}$	$\mathbf{P_b}$
–1	1	4	2	–4

		$\mathbf{m_{total}}$	$\mathbf{E_{total}}$	$\mathbf{P_{total}}$
		6	**6**	0

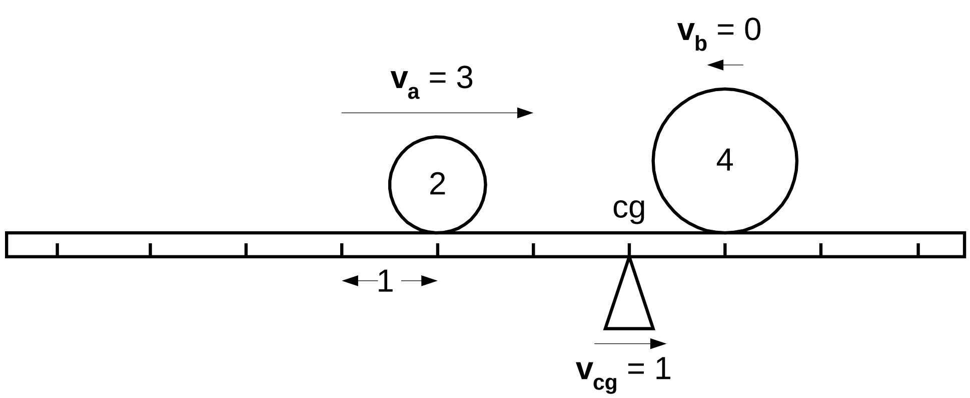
$\mathbf{E_{cg}} = \frac{1}{2}\, \mathbf{m_{total}}\, \mathbf{v_{cg}}^2 = \mathbf{0}$

and $\mathbf{P_{total}} = \mathbf{m_{total}}\, \mathbf{v_{cg}}$, so $\mathbf{v_{cg}} = \mathbf{P_{total}} / \mathbf{m_{total}} = 0$.

This velocity $\mathbf{v_{cg}}$ agrees with our intuitive sense.

Example 2

Choose a new reference frame so the whole *system* in Example 1 is moving to the right at a velocity of 1. Now $\mathbf{v_{cg}} = 1$, $\mathbf{v_a} = 3$, and $\mathbf{v_b} = 0$. Again, we'll find the energy and momentum of each particle and of the center of gravity.



Reference Frame 2:

$\mathbf{v}_a$	$\mathbf{v}_a^2$	$\mathbf{m}_a$	$\mathbf{E}_a$	$\mathbf{P}_a$
3	9	2	9	6

$\mathbf{v}_b$	$\mathbf{v}_b^2$	$\mathbf{m}_b$	$\mathbf{E}_b$	$\mathbf{P}_b$
0	0	4	0	0

		$\mathbf{m}_{total}$	$\mathbf{E}_{total}$	$\mathbf{P}_{total}$
		6	**9**	6

$\mathbf{E}_{cg} = ½\, \mathbf{m}_{total}\, \mathbf{v}_{cg}^2 = \mathbf{3}$

$\mathbf{v}_{cg} = \mathbf{P}_{total} / \mathbf{m}_{total} = 1$,

and the calculated velocity $\mathbf{v}_{cg}$ agrees with our picture.

Now we'll look at the energies. If the motion of the particles inside the group is hidden from our view, then we think the energy of the whole group is $\mathbf{E}_{cg}$. Call the hidden energy of the particles the "internal energy" $\mathbf{E}_{internal}$. Does $\mathbf{E}_{internal}$ change when the whole system moves? Well, the total energy is the sum of the particle energies:

$\mathbf{E}_{total} = \mathbf{E_a} + \mathbf{E_b}$

But it must also be true that

$\mathbf{E}_{total} = \mathbf{E}_{cg} + \mathbf{E}_{internal}$,

So we can calculate the internal energy by

$\mathbf{E}_{internal} = \mathbf{E}_{total} – \mathbf{E}_{cg}$

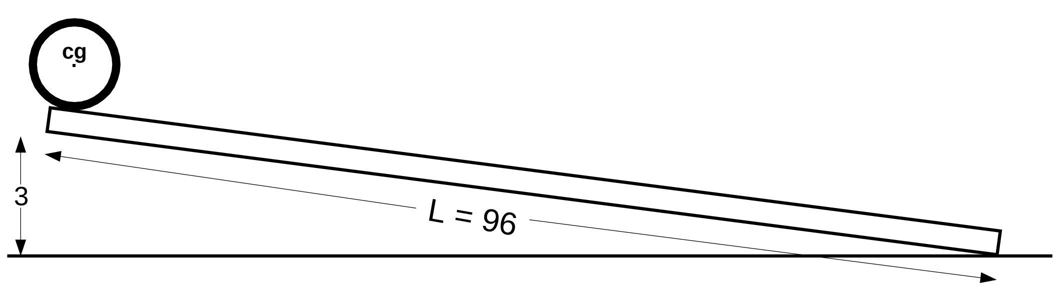
For Reference Frame 1: $\mathbf{E}_{internal} = 6 – 0 = 6$

For Reference Frame 2: $\mathbf{E}_{internal} = 9 – 3 = 6$

So $\mathbf{E}_{internal}$ is independent of whether the whole system (the center of gravity) moves or not. We can find $\mathbf{E}_{internal}$ by choosing a reference frame such that $\mathbf{E}_{cg} = 0$, so that $\mathbf{E}_{internal} = \mathbf{E}_{total}$.

Example 3

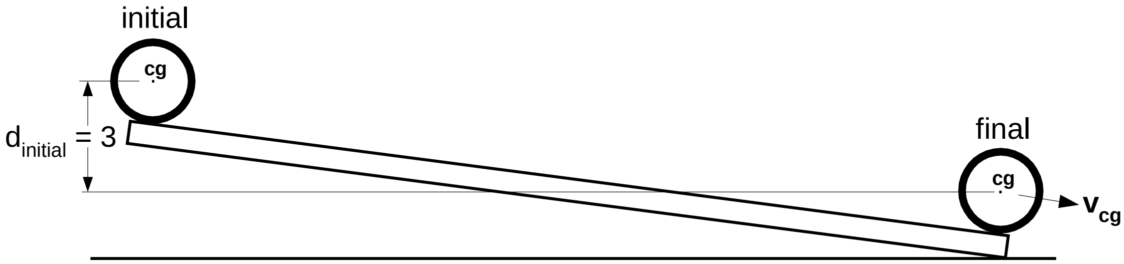
Let a tube with mass $\mathbf{m}_{total}$ roll down a ramp 96 inches long that is 3 inches higher on the left end.



The tube starts with some potential energy $\mathbf{E}_P$. From Lesson 3 we have

$$\mathbf{E}_P = -\mathbf{g}\cdot\mathbf{m}_{total}\cdot\mathbf{d}_{initial}$$

where $\mathbf{g} = -386$ inches/sec/sec is the acceleration due to gravity. We measure $\mathbf{d}_{initial}$ from the final position of the cg (when the tube reaches the end of the ramp). $\mathbf{d}_{initial} = 3$.



Therefore

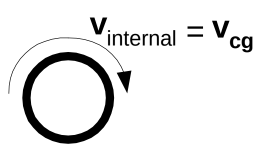
$$\mathbf{E}_P = -\mathbf{g}\cdot\mathbf{m}_{total}\cdot\mathbf{d}_{initial} = 386\cdot\mathbf{m}_{total}\cdot 3 = 1172\ \mathbf{m}_{total}.$$

When the tube reaches the "final" position, how has this potential energy been divided between internal energy (rotation) and kinetic energy of the center of gravity?

As the tube rolls, the velocity of its center of gravity (cg) increases, reaching $\mathbf{v}_{cg}$ at the final position. Then the final energy of the cg is

$$\mathbf{E}_{cg} = \tfrac{1}{2}\ \mathbf{m}_{total}\ \mathbf{v}_{cg}^2$$

But at the same time the cg is moving in a straight line, the tube is turning, giving it an internal energy. If we move our eye with the tube, the cg appears to be still, and the internal energy $\mathbf{E}_{internal}$ is the total energy of the tube's rotation. The speed of every particle of the tube's mass is moving at $\mathbf{v}_{internal} = \mathbf{v}_{cg}$.



Therefore the internal energy is

$$\mathbf{E}_{internal} = \tfrac{1}{2}\ \mathbf{m}_{total}\ \mathbf{v}_{internal}^2 = \tfrac{1}{2}\ \mathbf{m}_{total}\ \mathbf{v}_{cg}^2 = \mathbf{E}_{cg},$$

and the internal energy equals the cg energy. The total energy is sum of the two energies:

$$\mathbf{E}_{total} = \mathbf{E}_{cg} + \mathbf{E}_{internal} = \mathbf{E}_{cg} + \mathbf{E}_{cg} = 2\cdot\mathbf{E}_{cg}$$

$$\mathbf{E}_{cg} = \tfrac{1}{2}\ \mathbf{E}_{total}, \quad \mathbf{E}_{internal} = \tfrac{1}{2}\ \mathbf{E}_{total}$$

That is, the total energy is split equally between the internal energy and cg energy.

The final total kinetic energy equals the initial potential energy:

$$\mathbf{E}_{total} = \mathbf{E}_P = 1172\ \mathbf{m}_{total}.$$

$$\mathbf{E}_{cg} = \tfrac{1}{2}\ \mathbf{E}_{total} = 586\ \mathbf{m}_{total}.$$

Then we can solve for the final cg velocity from the cg energy:

$$\mathbf{E}_{cg} = \tfrac{1}{2}\ \mathbf{m}_{total}\ \mathbf{v}_{cg}^2$$

$$\mathbf{v}_{cg}^2 = 2\ \mathbf{E}_{cg} / \mathbf{m}_{total} = 1172.$$

Try $\mathbf{v}_{cg}$ = 35. But 35 × 35 = 1225, which is too high. Try $\mathbf{v}_{cg}$ = 34. And 34 × 34 = 1156, which is close enough. So we can say the final velocity is about $\mathbf{v}_{cg}$ = 34 in/sec.

We can check this velocity experimentally. We measure the time **T** that the tube takes to go from the initial position to the final position and find **T** = 5.6 sec. Since the ramp is 96 inches long, that means the average velocity is

$$\mathbf{v}_{ave} = 96 / 5.6 = 17 \text{ sec}$$

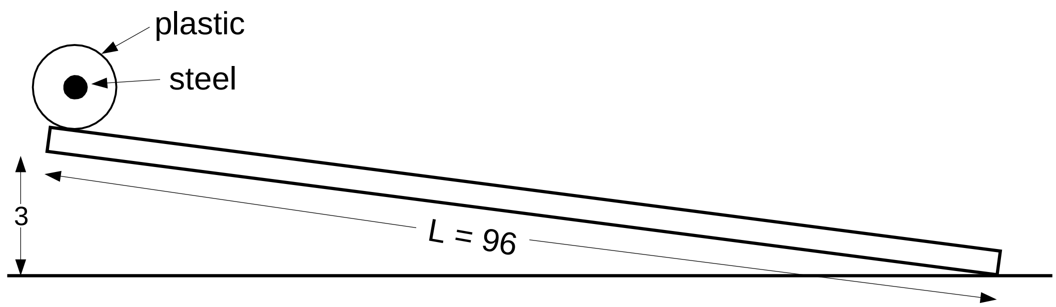
But from Lesson 2 we know that the final velocity is twice the average velocity:

$$\mathbf{v}_{cg} = 2 \cdot \mathbf{v}_{ave} = 34 \text{ in/sec}$$

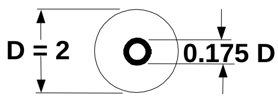
as we found from the potential energy.

Example 4

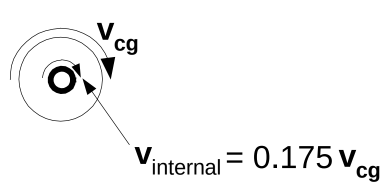
What if the mass $\mathbf{m}_{total}$ rolling down the ramp is almost all at the center of the tube? We can construct a special 2-inch tube out of light-weight plastic and position a ½-inch steel rod at its



center. The velocity of any particle of the steel mass is now much slower than the outer plastic shell. It can be shown that the steel rod acts like a steel tube of diameter 0.35 inch, which is 17.5% of the diameter of the plastic tube.



So the velocity of the equivalent steel tube is 17.5% of the velocity of the plastic tube:



But energies increase with the square of the velocity:

$$\mathbf{E}_{internal} = ½\, \mathbf{m}_{total}\, \mathbf{v}_{internal}^2$$

$$\mathbf{E}_{cg} = ½\, \mathbf{m}_{total}\, \mathbf{v}_{cg}^2$$

Since $0.175^2 = 0.03$,

$$\mathbf{E}_{internal} = 0.03\, \mathbf{E}_{cg},$$

The total energy is sum of the two energies:

$$\mathbf{E}_{total} = \mathbf{E}_{cg} + \mathbf{E}_{internal} = \mathbf{E}_{cg} + 0.03\ \mathbf{E}_{cg} = 1.03\ \mathbf{E}_{cg}$$

$$\mathbf{E}_{cg} = \mathbf{E}_{total}/1.03 = 0.97\ \mathbf{E}_{total}$$

$$\mathbf{E}_{internal} = 0.03\ \mathbf{E}_{cg} = 0.03(0.97\ \mathbf{E}_{total}) = 0.029\ \mathbf{E}_{total}$$

But

$$\mathbf{E}_{total} = \mathbf{E}_{P} = 1172\ \mathbf{m}_{total}$$

Therefore

$$\mathbf{E}_{cg} = 0.97\ \mathbf{E}_{total} = 0.97(1172\ \mathbf{m}_{total}) = 1137\ \mathbf{m}_{total}$$

Then we can solve for the final cg velocity from the cg energy:

$$\mathbf{E}_{cg} = ½\ \mathbf{m}_{total}\ \mathbf{v}_{cg}^{2}$$

$$\mathbf{v}_{cg}^{2} = 2\ \mathbf{E}_{cg} / \mathbf{m}_{total} = 2(1137) = 2274.$$

Try $\mathbf{v}_{cg}$ = 48. But 48 × 48 = 2304, which is too high. Try $\mathbf{v}_{cg}$ = 47. But 47 × 47 = 2209, which is too low. So the final velocity is about halfway between: $\mathbf{v}_{cg}$ = 47.5 in/sec.

We can check this velocity experimentally. We measure the time **T** that the tube takes to go from the initial position to the final position and find **T** = 4.0 sec. Since the ramp is 96 inches long, that means the average velocity is

$$\mathbf{v}_{ave} = 96 / 4.0 = 24\ \text{sec}$$

But the final velocity is twice the average velocity:

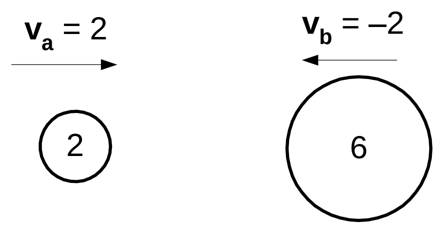
$$\mathbf{v}_{cg} = 2 \cdot \mathbf{v}_{ave} = 48\ \text{in/sec}$$

which is about the same velocity we found from the potential energy.

Problems

Problem 1

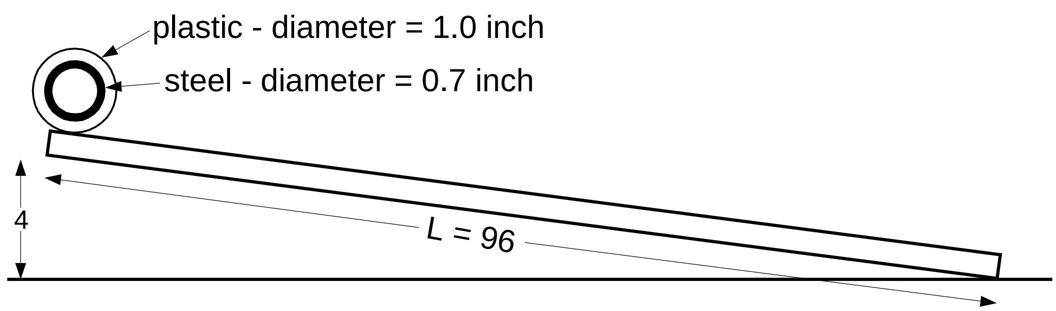
Two balls with masses and velocities shown below are approaching each other.



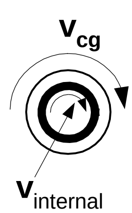
Find the energy and momentum of each ball and of the center of gravity. (Use a table as in Example 1.) Find $\mathbf{E}_{total}$, $\mathbf{E}_{internal}$, and $\mathbf{v}_{cg}$.

Problem 2

A ramp of length 96 inches is raised a height **d** = 4 inches at the left end. At that end is a light-weight plastic tube of diameter 1 inch with a steel tube of diameter 0.7 inch and mass $\mathbf{m}_{total}$ suspended at its center. Find the potential energy $\mathbf{E}_P$ of the steel tube (compared to its final position at the right end of the board) in terms of $\mathbf{m}_{total}$.



The rotational velocity of the plastic tube will be $\mathbf{v}_{cg}$ when the tube rolls. Find the velocity $\mathbf{v}_{internal}$ of the steel tube in terms of $\mathbf{v}_{cg}$.

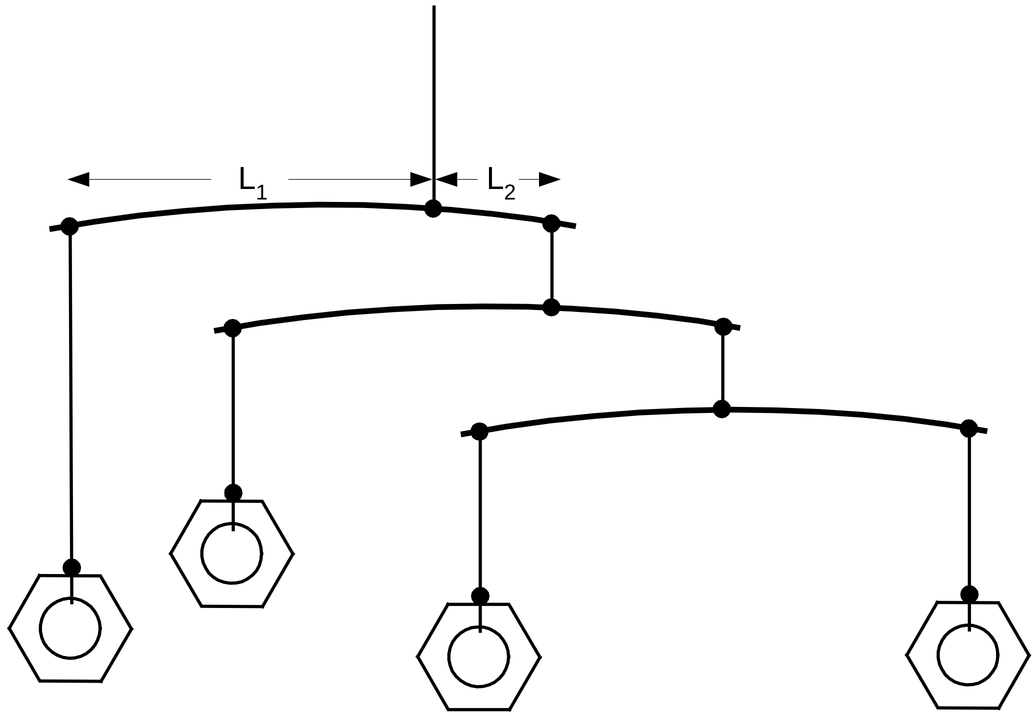


Find the internal energy $\mathbf{E}_{internal}$ in terms of the center-of-gravity energy $\mathbf{E}_{cg}$. Find $\mathbf{E}_{internal}$ in terms of the total energy $\mathbf{E}_{total}$, where $\mathbf{E}_{total} = \mathbf{E}_{internal} + \mathbf{E}_{cg}$. Since $\mathbf{E}_{total} = \mathbf{E}_P$, solve for the final cg velocity $\mathbf{v}_{cg}$. (This is similar to Example 4.)

Experiments

Experiment 1

Construct a system of particles (machine nuts) suspended from stiff wires by strings as shown:

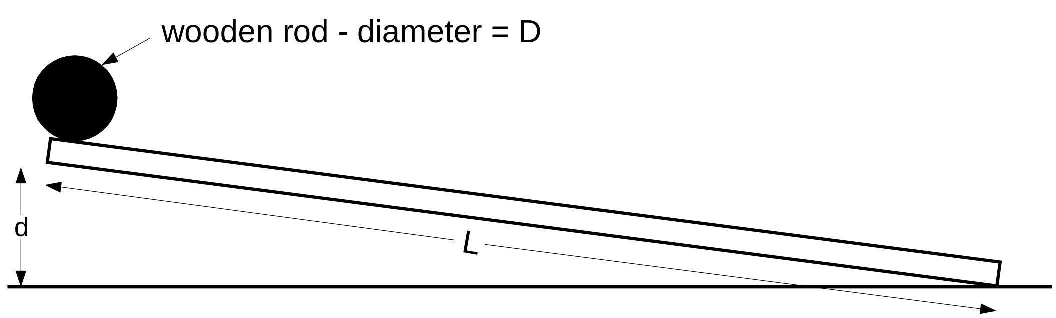


You'll have to adjust the position of the knot that supports each wire so the wire balances. Then measure the distances from the knot to each end, and find the ratio L_1/L_2, where L_1 is the longer distance. Can you explain each of the three ratios?

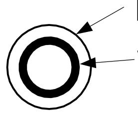
Try blowing on the nuts to give the system some internal energy. Holding the top string, move the center of gravity (cg) around and see if the internal motion has any effect on the motion of the cg.

Experiment 2

Put a board of length **L** on the floor, and put a thin book under the right end so the board is level. Then put a fat book under the left end to raise it some height **d**. Put a wooden rod of diameter **D** and mass $\mathbf{m}_{total}$ on the left end. Find the potential energy $\mathbf{E}_P$ of the rod (compared to its final position at the right end of the board) in terms of its mass.



It can be shown that the mass of the rod acts like all its mass is at a radius 0.35·**D** from the center of gravity. That is, it acts like a wooden tube of diameter 0.7·**D** and mass $\mathbf{m}_{total}$ suspended in a light-weight plastic tube of diameter **D**. (This is the same structure as in Problem 2.)



Use the same method as in Problem 2 to solve for the final center-of-gravity velocity $\mathbf{v}_{cg}$.

Time how long it takes for the rod to roll the length of the ramp. What is the average velocity $\mathbf{v}_{ave}$? What is the final velocity $\mathbf{v}_{cg}$? Does this agree with the velocity you calculated from the potential energy?

Experiment 3

Roll a wooden ball down the ramp in Experiment 2. Is the time it takes to roll the length of the ramp longer or shorter than the time for the wooden rod? Explain.