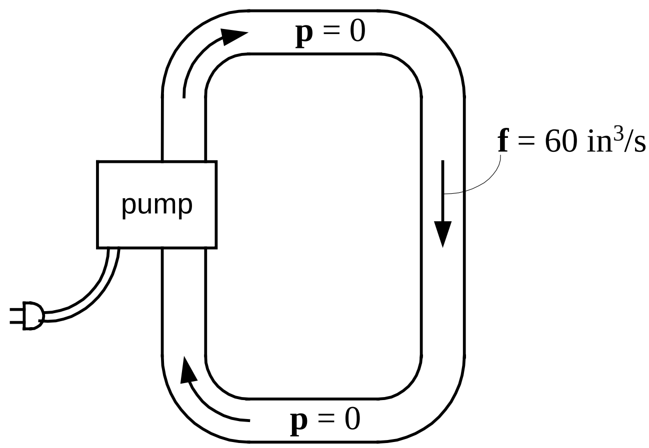
Chapter 10
Electrical Power

Mechanical power and force is easy to get a feel for because we interact with it every day. But we don't interact with electrical power and voltage much, and if we do, it's usually an unpleasant experience. Fortunately, electrical systems are quite analogous to mechanical systems, so we'll start out by looking at pumps and water.

Example 1

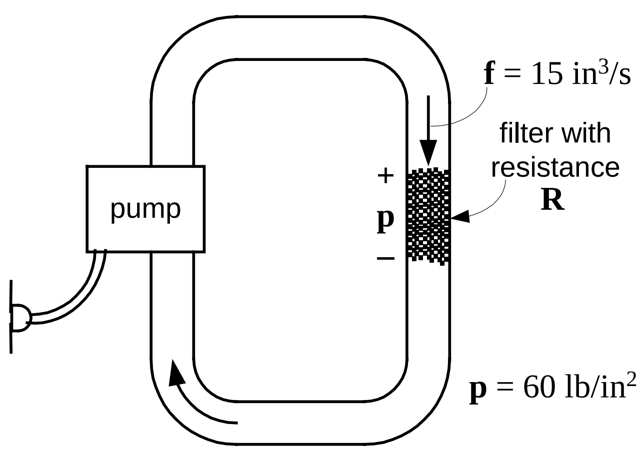
Consider the system below, where a pump causes water to circulate around a closed pipe lying on the ground (the picture is a top view). This is similar to the system in Lesson 9, but now no water is added to or lost from the system. The flow rate **f** of the water is 60 cubic inches per second. What pressure is required to sustain this flow?

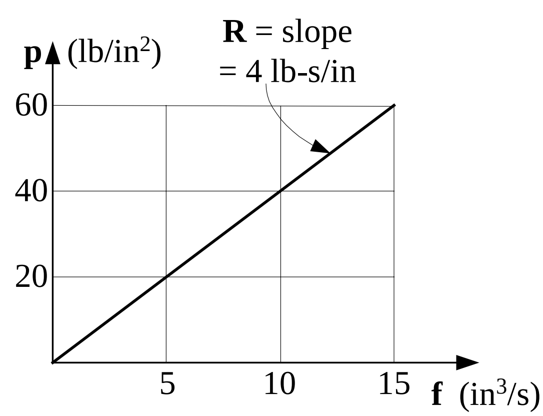


Except for some friction (which we neglect), no pressure is needed to keep the water circulating. So the pump doesn't require any power; it's unplugged. The pressure everywhere in the pipe is **p** = 0.

Example 2

The system shown below is a little more interesting; it has a filter for cleaning the water, and the filter resists the flow of water. Therefore some pressure difference **p** is required across the filter





to make water through it at a certain rate. Let's say the flow rate is **f** = 15 in^3/s. The pressure required for this flow rate depends on the resistance **R** of the filter. For our case the filter requires a pressure difference **p** of 4 lb/in^2 for every cubic inch per second of flow.

$$R = 4\frac{lb/in^2}{in^3/s} = 4\frac{lb\text{-}s}{in}.$$

For a given resistance, the pressure **p** is proportional to the flow rate **f**, as shown in the plot of **p** and **f**. Therefore the pressure is given by

$$\mathbf{p} = \mathbf{R}\cdot\mathbf{f} = (4\ lb\text{-}s/in)\cdot(15\ in^3/s) = 60\ lb/in^2.$$

What power **P** is the pump putting out? We know that

$$\mathbf{P} = \mathbf{F}\cdot\mathbf{v}$$

where **F** is the force the pump is applying to the water, and **v** is the velocity of the water. We can find both **F** and **v** from **p** and **f** if we know the cross-section area **A** of the pipe:

$$\mathbf{F} = \mathbf{p}\cdot\mathbf{A}, \text{ and } \mathbf{v} = \mathbf{f}/\mathbf{A}.$$

Therefore the power is $\mathbf{P} = \mathbf{F}\cdot\mathbf{v} = (\mathbf{p}\cdot\mathbf{A})\cdot(\mathbf{f}/\mathbf{A}) = \mathbf{p}\cdot\mathbf{f}$

so $\mathbf{P} = \mathbf{p}\cdot\mathbf{f}$

(It's a good thing the area **A** disappeared because the pump doesn't worry about the pipe's cross-section area—just the pressure and flow rate.) In our case,

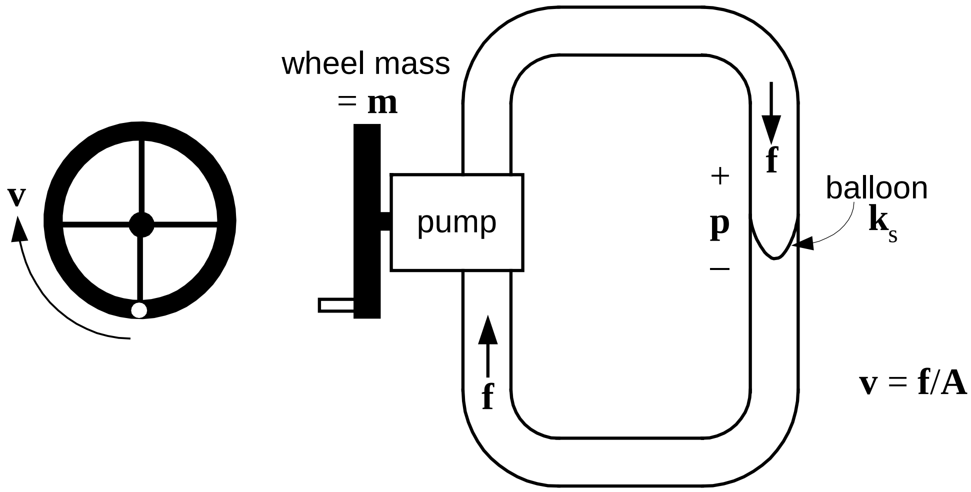
$$\mathbf{P} = (60\ lb/in^2)\cdot(15\ in^3/s) = 900\ in\text{-}lb/s$$

or $\mathbf{P} = (900\ in\text{-}lb/s)/(12\ in/ft) = 75\ ft\text{-}lb/s.$

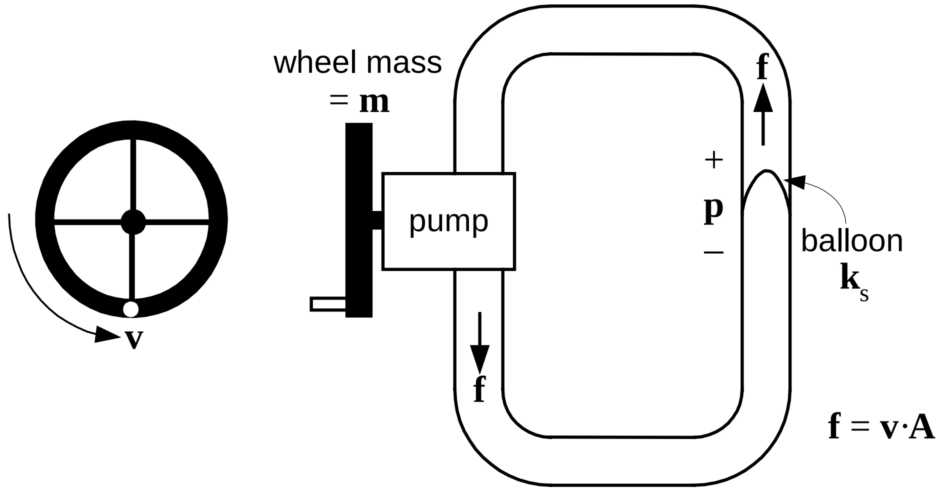
The electrical power (in watts) to run the pump is

$$\mathbf{P} = (75\ ft\text{-}lb/s)\cdot(1.356\ watt/(ft\text{-}lb/s)) = 102\ watts.$$

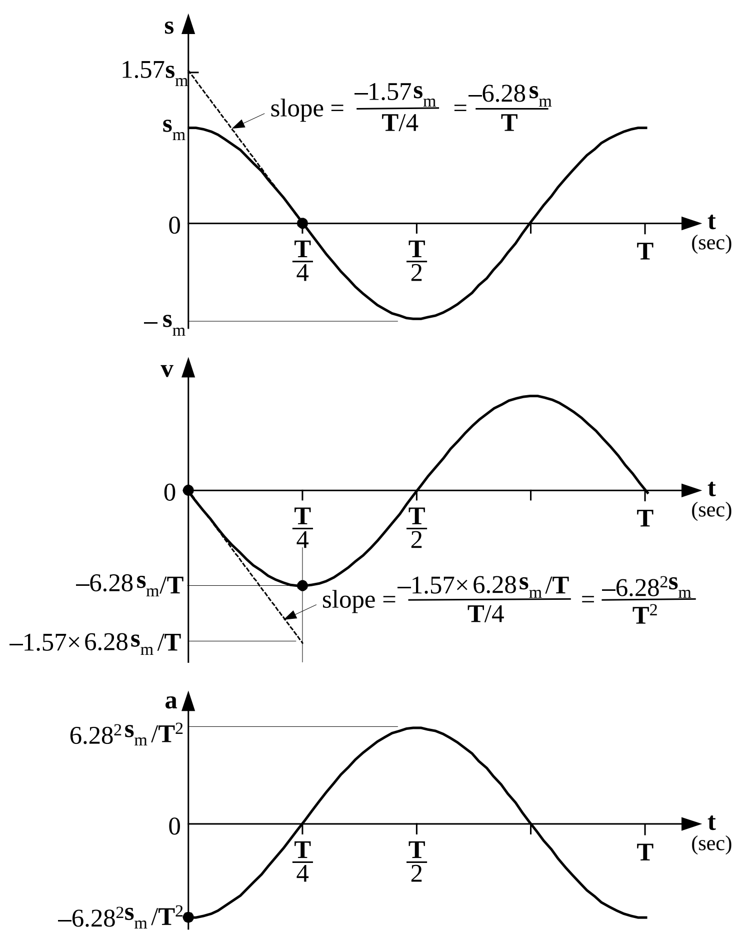
Example 3



The pump motor in Example 2 is replaced with a flywheel with a handle on it. The filter (with resistance) is replaced with a balloon stretched across the pipe. Using the handle, we turn the wheel at velocity **v**, causing the water in the pipe to move with velocity **v** (where **v** = **f**/**A**). The water flow **f** stretches the rubber sheet, increasing pressure **p**. We let go of the handle, but the wheel's mass **m** keeps it turning until increasing pressure **p** (as the balloon stretches) slows the wheel. The pressure finally reverses the wheel's motion, and the balloon stretches in the other direction, producing a negative **p** (see below).



This cycle repeats, transferring energy back and forth between the flywheel and the balloon. When the water is moving, the energy is stored in the motion of the wheel, and when the water stops, the energy is stored in the stretched balloon. The result is harmonic oscillation like we saw in Lesson 7 (see plots below). The distance the balloon stretches is **s** (with a maximum of $\mathbf{s}_m$). The velocity



of the water (and of the wheel) is $\mathbf{v}$. The acceleration of the wheel is $\mathbf{a}$, with a maximum value of $\mathbf{a}_m = 6.28^2\mathbf{s}_m/\mathbf{T}^2$ at the time that $\mathbf{s} = -\mathbf{s}_m$. If we let a dot over a variable indicate the rate of change (the slope) of the variable, then we can write

$$\mathbf{v} = \dot{\mathbf{s}},$$

and

$$\mathbf{a} = \dot{\mathbf{v}}.$$

As in Lesson 7, we find the period $\mathbf{T}$ by setting the force $\mathbf{F} = \mathbf{m}\cdot\mathbf{a}$ on the mass equal to the force $\mathbf{F} = \mathbf{s}/\mathbf{k}_s$ of the balloon stretching. The balloon's "spring constant" is $\mathbf{k}_s$. The equation is the same as before:

$$\mathbf{T}^2 = 6.28^2\cdot\mathbf{m}\cdot\mathbf{k}_s, \quad \mathbf{T} = 6.28\cdot\sqrt{\mathbf{m}\cdot\mathbf{k}_s}$$

The variables $\mathbf{F}$ and $\mathbf{v}$ are related to the pressure $\mathbf{p}$ and flow rate $\mathbf{f}$ by

$$\mathbf{F} = \mathbf{p}\cdot\mathbf{A} \ \text{ and } \ \mathbf{v} = \mathbf{f}/\mathbf{A},$$

where $\mathbf{A}$ is the cross-section area of the pipe. Then we can write the equation for the mass as

$$\mathbf{F} = \mathbf{m}\cdot\dot{\mathbf{v}} \ \text{ or } \ \mathbf{p} = \mathbf{m}\cdot\dot{\mathbf{f}}/\mathbf{A}^2$$

and the equation for the balloon is

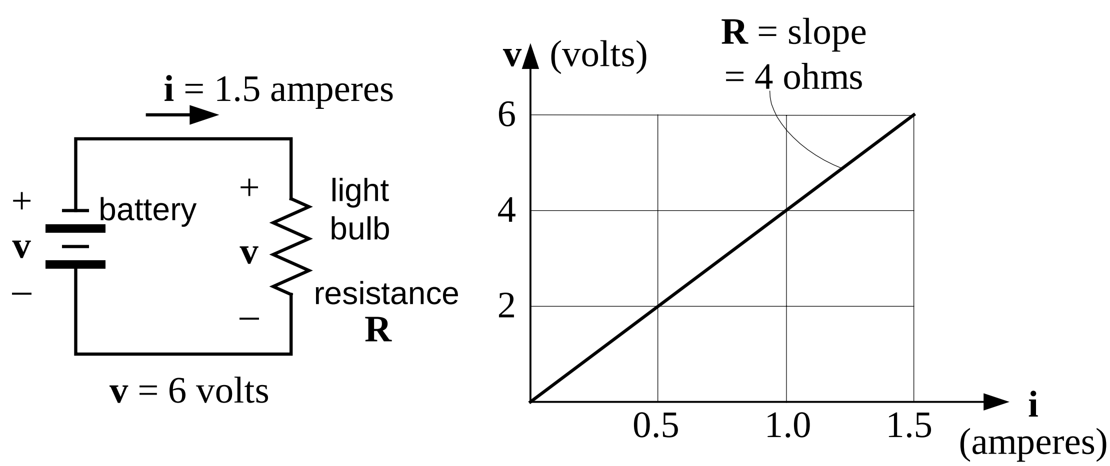
$$\mathbf{F} = \mathbf{s}/\mathbf{k}_s \ \text{ or } \ \dot{\mathbf{F}} = \dot{\mathbf{s}}/\mathbf{k}_s = \mathbf{v}/\mathbf{k}_s$$

or

$$\dot{\mathbf{p}} = \mathbf{f}/(\mathbf{k}_s\mathbf{A}^2)$$

Example 4

The electrical circuit of a flashlight (see below) is very much like the mechanical system in Example 2. A battery provides the voltage $\mathbf{v}$ (electrical pressure) to force a current $\mathbf{i}$ (an electrical flow) of electrons through the light bulb (with resistance $\mathbf{R}$).



The voltage is $\mathbf{v} = 6$ volts, and the current $\mathbf{i}$ is proportional to the voltage, depending on the resistance $\mathbf{R}$ of the bulb.

$$\mathbf{i} = \mathbf{v}/\mathbf{R}.$$

In our case, $\mathbf{R} = 4$ ohms, so $\mathbf{i} = (6 \text{ volts})/(4 \text{ ohms}) = 1.5$ amperes. The voltage $\mathbf{v}$ acts like the pressure $\mathbf{p}$ in Example 2, and the current $\mathbf{i}$ acts like the flow $\mathbf{f}$.

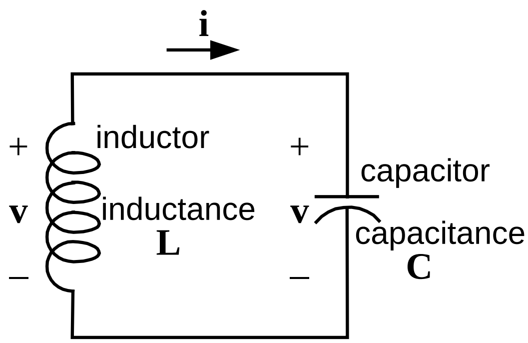
The power the battery provides is given by

$$\mathbf{P} = \mathbf{v} \cdot \mathbf{i} = (6 \text{ volts}) \cdot (1.5 \text{ amperes}) = 9 \text{ watts.}$$

Note the similarity to $\mathbf{P} = \mathbf{p} \cdot \mathbf{f}$ in Example 2. If the battery's voltage were reduced to 2 volts, the current would reduce to 0.5 amperes (see the plot above), and the power would be $\mathbf{P}$ = (2 volts)·(0.5 ampere) = 1 watt.

Example 5

The electrical circuit below has an inductor, which is a coil of wire, and a capacitor, which is two metal plates close together. As we force current **i** through the inductor, the magnetic field that forms around the coil opposes the increase in current. But once the current is flowing, when it tries to stop, the collapsing field opposes any reduction in current. This is similar to the massive wheel on the pump in Example 3, where the inertia of the mass opposes any change in the flow.



The electrical "mass" of the inductor is its inductance **L**. Let the symbol $\dot{\mathbf{i}}$ represent the rate of change (the slope) of the current **i**. Then

$$\mathbf{v} = -\mathbf{L} \cdot \dot{\mathbf{i}}.$$

Compare this with the equation for the mass in Example 3:

$$\mathbf{p} = (\mathbf{m}/\mathbf{A}^2) \cdot \dot{\mathbf{f}}$$

We see that inductance **L** is like mass **m** (scaled by $\mathbf{A}^2$), voltage **v** is like pressure **p**, and current **i** is like flow **f**.

The electrical current (flow of electrons) can't pass through the capacitor because of the gap. But increasing voltage forces electrical charge onto the top metal plate, and the increasing charge repels charge from the bottom plate, causing some current to flow. When the voltage stops increasing, the current stops. This is similar to Example 3 where pressure increase stretches the balloon and causes some water movement.

The electrical "stretchiness" of the capacitor is its capacitance **C**. Let the symbol $\dot{\mathbf{v}}$ represent the rate of change (the slope) of the voltage **v**. Then

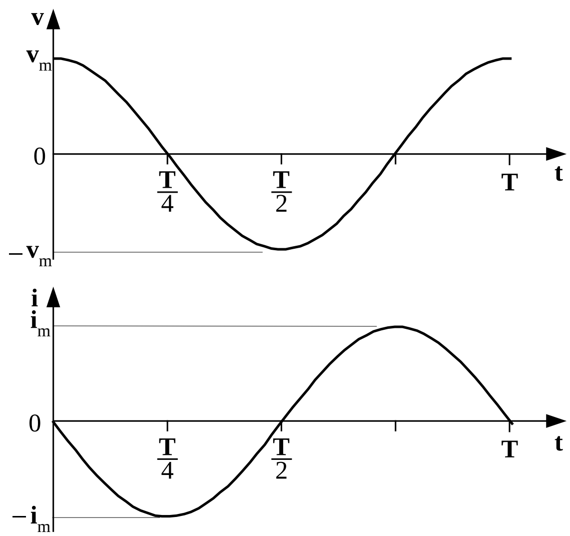
$$\dot{\mathbf{v}} = \mathbf{i}/\mathbf{C}.$$

Compare this with the equation for the balloon in Example 3:

$$\dot{\mathbf{p}} = \mathbf{f}/(\mathbf{k}_s\mathbf{A}^2)$$

We see that capacitance **C** is like the spring constant $\mathbf{k}_s$ (scaled by $\mathbf{A}^2$), voltage **v** is like pressure **p**, and current **i** is like flow **f**.

The L-C circuit is a harmonic oscillator, with electrons sloshing back and forth from one capacitor plate to the other, the inductor providing the momentum to keep the current going.



The voltage and current plots are sine waves. Through an analysis similar to that in Example 3, we can show the period of oscillation is given by

$$\mathbf{T}^2 = 6.28^2\cdot\mathbf{L\cdot C}, \qquad \mathbf{T} = 6.28\cdot\sqrt{\mathbf{L\cdot C}}$$

For example if **L** = 0.01 henry and **C** = 0.0001 farad, then

$$\mathbf{T} = 6.28\cdot\sqrt{\mathbf{L\cdot C}} = 6.28\cdot\sqrt{0.01\cdot 0.0001} = 6.28\cdot 0.001 = 0.00628 \text{ seconds.}$$

This is a frequency of $1/\mathbf{T} = 1/0.00628 = 160$ hertz.

Suppose the maximum voltage on the capacitor is $\mathbf{v}_m = 5$ volts. We can show from the maximum slope of **v** in the plot and from $\dot{\mathbf{v}} = \mathbf{i}/\mathbf{C}$ that the maximum current is $\mathbf{i}_m = 0.5$ amperes.

Problems

Problem 1

Suppose the pump in Example 2 is not so powerful and produces a flow only half as great: $\mathbf{f} = 7.5\ \text{in}^3/\text{s}$. Find the new pressure **p** and new power **P**. Is the pump half as powerful as when it produced **P** = 900 in-lb/s ?

Problem 2

The battery voltage in Example 4 is reduced to half: **v** = 3 volts. Find the new current **i** and the new power **P**. Is the battery providing half the **P** = 9 watts it provided for **v** = 6 volts?

Problem 3

What is the maximum acceleration $\mathbf{a}_m$ (in terms of $\mathbf{s}_m$ and **T)** shown by the plot in Example 3? Set $\mathbf{F} = \mathbf{m}\cdot\mathbf{a}_m$ equal to $\mathbf{F} = \mathbf{s}_m/\mathbf{k}_s$ and solve for $\mathbf{T}^2$.

Problem 4

We can find the period **T** in Example 5 from the inductor equation $\mathbf{v} = -\mathbf{L}\cdot\ddot{\mathbf{i}}$ and the capacitor equation $\dot{\mathbf{v}} = \mathbf{i}/\mathbf{C}$.

Use the plot for $\mathbf{v}$ to find the slope $\dot{\mathbf{v}}$ of $\mathbf{v}$ at $\mathbf{t} = \mathbf{T}/4$ in terms of $\mathbf{v}_m$. Then use the capacitor equation to find $-\mathbf{i}_m$ in terms of $\mathbf{v}_m$.

Use the plot for $\mathbf{i}$ to find the slope $\dot{\mathbf{i}}$ of $\mathbf{i}$ at $\mathbf{t} = 0$ in terms of $\mathbf{i}_m$. Then use the inductor equation to find $-\mathbf{v}_m$ in terms of $\mathbf{i}_m$.

What does $\mathbf{T}^2$ have to be for these two results to be the same?

Problem 5

Using the plot for $\mathbf{v}$ in Example 5, Find the slope $\dot{\mathbf{v}}$ of $\mathbf{v}$ for $\mathbf{t} = \mathbf{T}/4$ if $\mathbf{v}_m = 6$ volts. Then use the capacitor equation to find the value of $\mathbf{i}$ for $\mathbf{t} = \mathbf{T}/4$.

Problem 6

The power stored in an inductor's magnetic field is given by $\mathbf{P}_L = ½\cdot\mathbf{L}\cdot\mathbf{i}^2$. The power stored in a capacitor's electric field is given by $\mathbf{P}_C = ½\cdot\mathbf{C}\cdot\mathbf{v}^2$. Find $\mathbf{P}_L$ for $\mathbf{i}$ equals the $\mathbf{i}_m$ in Problem 5. Find $\mathbf{P}_C$ for $\mathbf{v}$ equals the $\mathbf{v}_m$ in Problem 5. From the plots for $\mathbf{v}$ and $\mathbf{i}$ in Example 5, sketch plots for $\mathbf{P}_L$ and $\mathbf{P}_C$. Does $\mathbf{P}_L + \mathbf{P}_C$ stay constant?