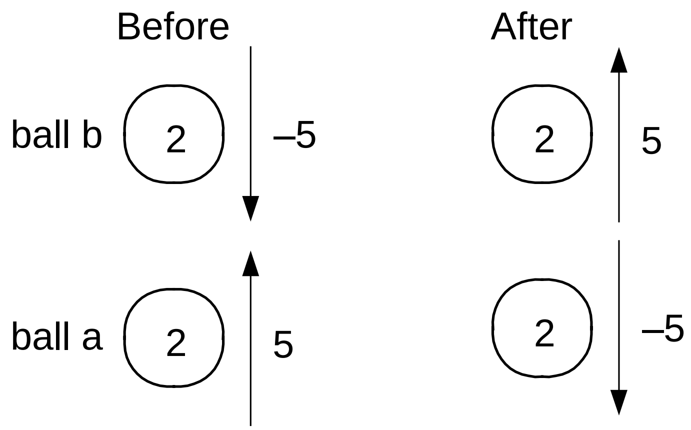
Chapter 1
Kinetic Energy

Kinetic energy: $E = ½ \times m \times v \times v = ½\ mv^2$, where m is the mass and v is the velocity.
Physical law: The total energy must be conserved.

Example 1



Each ball has mass 2. We're not saying whether it's 2 pounds or 2 grams. The behavior will be the same so long as the ratio of the two masses is specified (they're equal here). The velocities of the balls both have magnitude 5. (We say velocity is positive when it's upward and negative when it's downward.) We don't specify 5 inches per second or 5 miles per hour; the important thing is that they are equal and opposite.

From the symmetry of this experiment we can determine the velocities after collision (picture on the right). Since the balls come together at the same speeds, they bounce and leave each other at the same speeds (symmetrically, like the "Before" picture). The speeds must be 5 so energy is conserved. Note the event looks the same if you run the movie backwards (time goes backward).

We'll show the total energy E is the same after as before the collision:

Before: $E_a = ½\ m_a v_a^2 = ½ \times 2 \times 5^2 = ½ \times 2 \times 25 = 25$

$E_b = ½\ m_b v_b^2 = ½ \times 2 \times (-5)^2 = ½ \times 2 \times 25 = 25$

$E = E_a + E_b = 25 + 25 = \mathbf{50}$

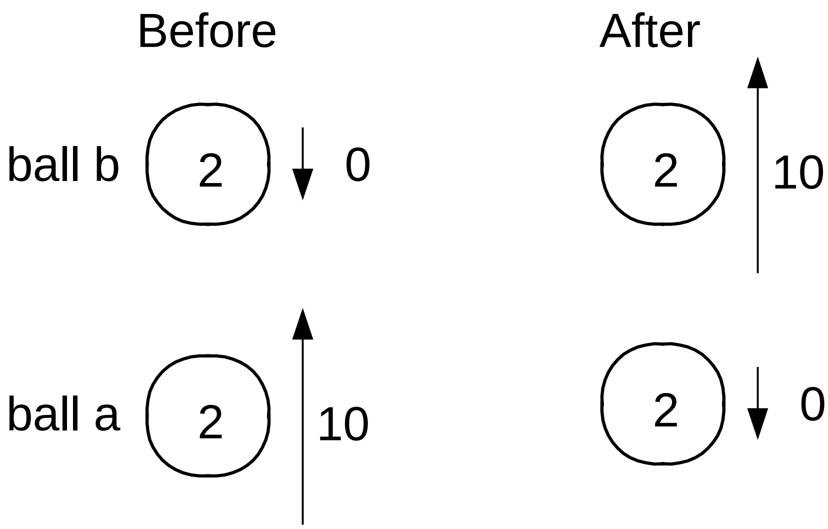
After: $E_a = ½\ m_a v_a^2 = ½ \times 2 \times 5^2 = ½ \times 2 \times 25 = 25$

$E_b = ½\ m_b v_b^2 = ½ \times 2 \times (-5)^2 = ½ \times 2 \times 25 = 25$

$E = E_a + E_b = 25 + 25 = \mathbf{50}$

Example 2: Changing our frame of reference (point of view)

Suppose I move my head downward with a velocity of –5, so it's following ball **b** in Example 1 before collision. Then from my frame of reference ball **b** has a velocity of 0. Similarly, ball **a** is moving toward me (before collision) as I'm moving toward it, so ball **a** has a velocity of 10 in my frame of reference. (See "Before" picture below.)



Look at the "After" picture in Example 1 on the previous page, and change our frame of reference for it too. I'm moving my head downward with a velocity of –5, so it's following ball **a**. Then from my frame of reference ball **a** has a velocity of 0. Similarly, ball **b** is moving toward me as I'm moving toward it, so ball **b** has a velocity of 10 in my frame of reference. (See "After" picture above.)

So from the new point of view ball **a** hits ball **b** which is sitting still. After the collision, ball **a** has come to a stop, and ball **b** is moving as fast as ball **a** was. Let's show that the total energy doesn't change.

Before: $E_a = ½\ m_a v_a^2 = ½ \times 2 \times 10^2 = ½ \times 2 \times 100 = 100$

$E_b = ½\ m_b v_b^2 = ½ \times 2 \times 0^2 = 0$

$E = E_a + E_b = 100 + 0 = \mathbf{100}$

After: $E_a = ½\ m_a v_a^2 = ½ \times 2 \times 0^2 = 0$

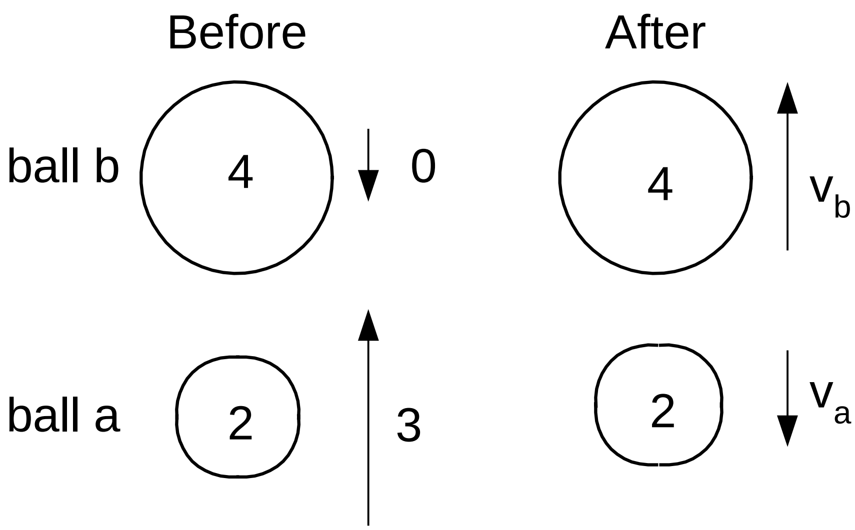
$E_b = ½\ m_b v_b^2 = ½ \times 2 \times 10^2 = ½ \times 2 \times 100 = 100$

$E = E_a + E_b = 0 + 100 = \mathbf{100}$

Even though the total energy is greater than in Experiment 1, it's still true the total energy is the same (100) before and after the collision.

Example 3

Now we'll make the balls have different masses; ball **b** has twice the mass of ball **a**. Let ball **b** be initial still (velocity 0) and ball **a** have velocity 3.



Knowing the "before" velocities 3 and 0, we want to calculate the "After" velocities v_a and v_b. We don't have symmetry to help use here; we have to calculate. To find the two unknown velocities we need two equations. But we have just one—the equation that says the energy after is the same as the energy before.

Before: $E_a = ½\ m_a v_a^2 = ½ \times 2 \times 3^2 = ½ \times 2 \times 9 = 9$

$E_b = ½\ m_b v_b^2 = ½ \times 4 \times 0^2 = 0$

$E = E_a + E_b = 9 + 0 = \mathbf{9}$

After: $E_a = ½\ m_a v_a^2 = ½ \times 2 \times v_a^2 = v_a^2$

$E_b = ½\ m_b v_b^2 = ½ \times 4 \times 10^2 = 2\ v_b^2$

$E = E_a + E_b = \mathbf{v_a^2 + 2\ v_b^2}$

Setting the "before energy equal to the "After" energy, we get the equation

$\mathbf{9 = v_a^2 + 2\ v_b^2}$

We can get a second equation by changing our frame of reference, adding some number like 5 to all four velocities in the picture above. It can be shown that this is the same as using the "Conservation of Momentum" law for our second equation (See Appendix A). The momentum of an object (a ball here) is defined as

$P = m \times v.$

Before: $P_a = m_a v_a = 2 \times 3 = 6$

$P_b = m_b v_b = 4 \times 0^2 = 0$

$P = P_a + P_b = 6 + 0 = \mathbf{6}$

After: $P_a = m_a v_a = 2\ v_a$

$P_b = m_b v_b = 4\ v_b$

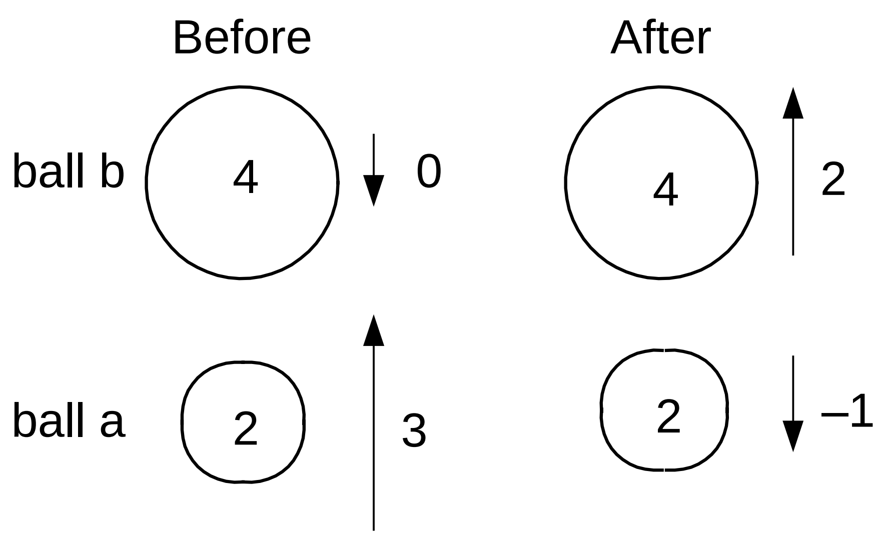
$E = E_a + E_b = \mathbf{2\ v_a + 4\ v_b}$

Setting the "Before" momentum equal to the "After" momentum, we get the equation

$\mathbf{6 = 2\ v_a + 4\ v_b}$

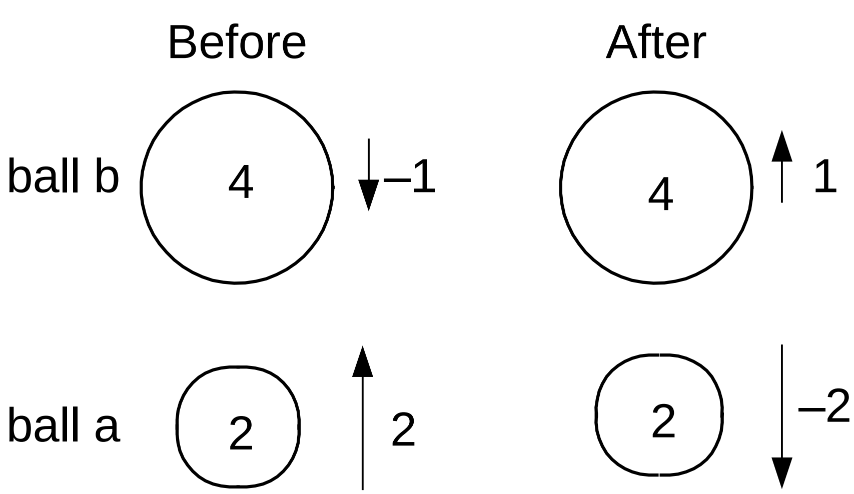
Solving this equation together with $\mathbf{9} = \mathbf{v_a^2 + 2\ v_b^2}$, we can get the values for v_b and v_b (see Appendix B):

$V_a = -1,\ \ v_b = 2.$



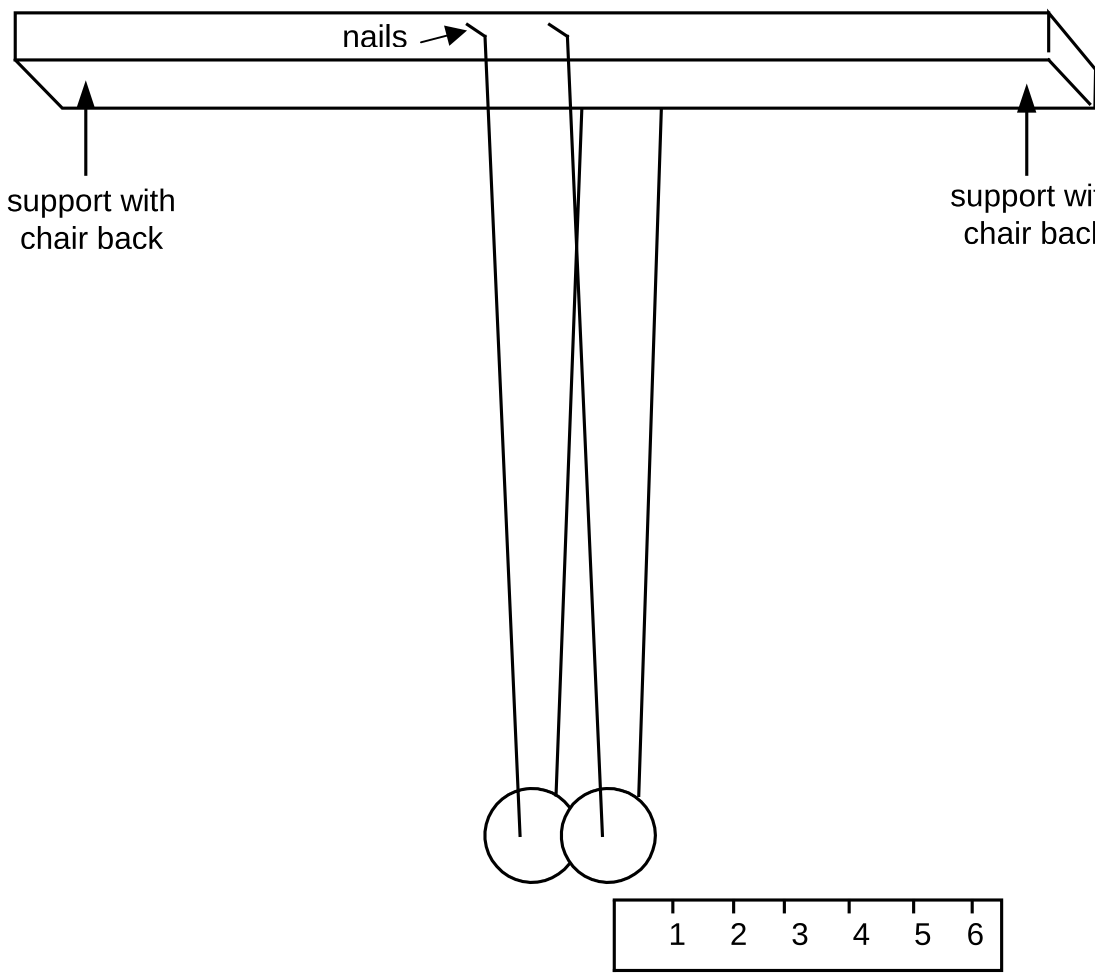
Example 4: Change our frame of reference (moving past the Example 3 experiment)

We want a new frame of reference so the "Before" and "After" velocities of ball **a** have the same magnitude. (I move my head downward with a velocity to reduce the 3 and the –1 by the same amount. Note that reducing –1 makes its magnitude larger.) If we subtract 1 from all four velocities (my head is moving upward with velocity 1), then 3 becomes **2** and –1 becomes **–2** (same magnitude), and 0 becomes **–1** and 2 becomes **1** (see picture below). Note that the velocity of ball **b** also has the same magnitude before and after now. (A coincidence, or will that always be true?)



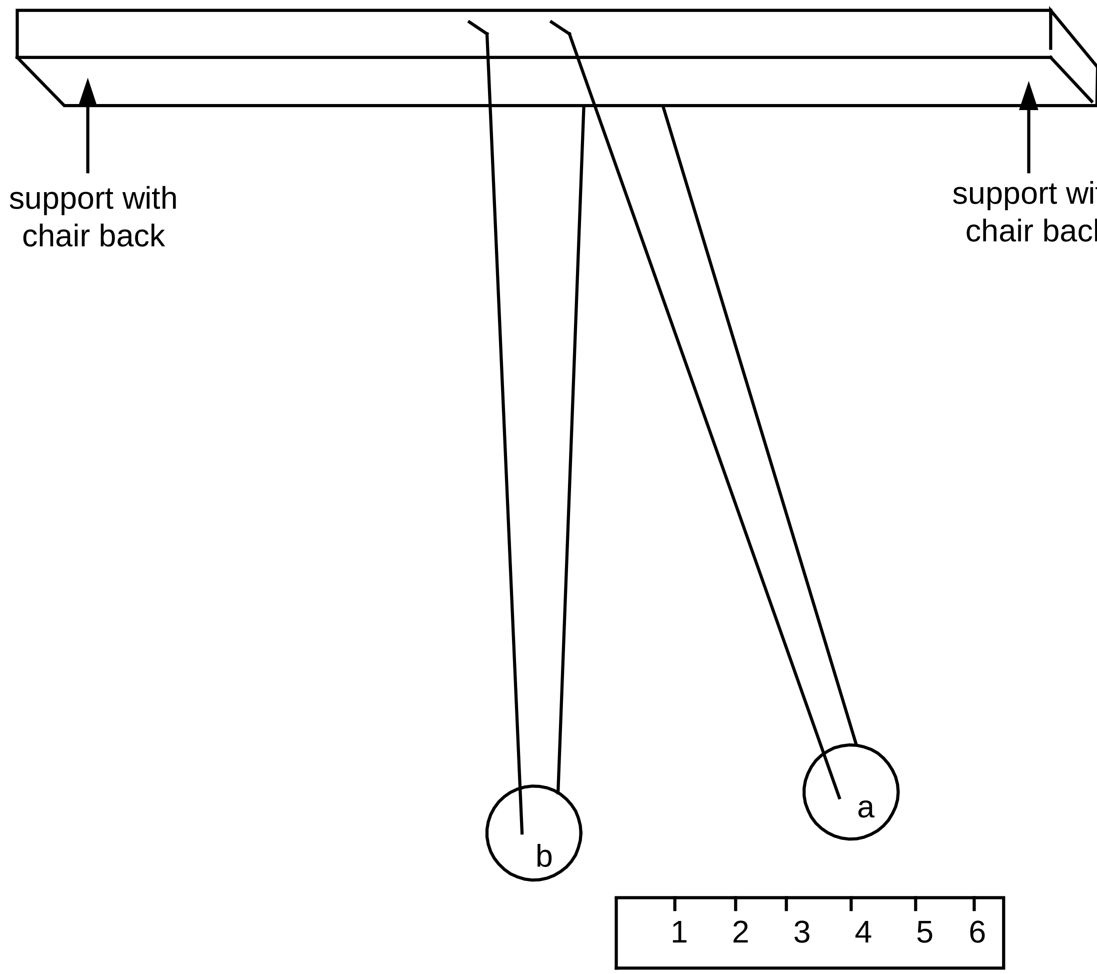
Experiments

Construct a dual pendulum as shown below. Position the four nails so the two balls just touch. The length of the strings is not too important; longer makes things slower, but the balls may not swing as straight. The ruler will be used to measure how far the pendulum is pulled back and how far it swings.



Experiment 1

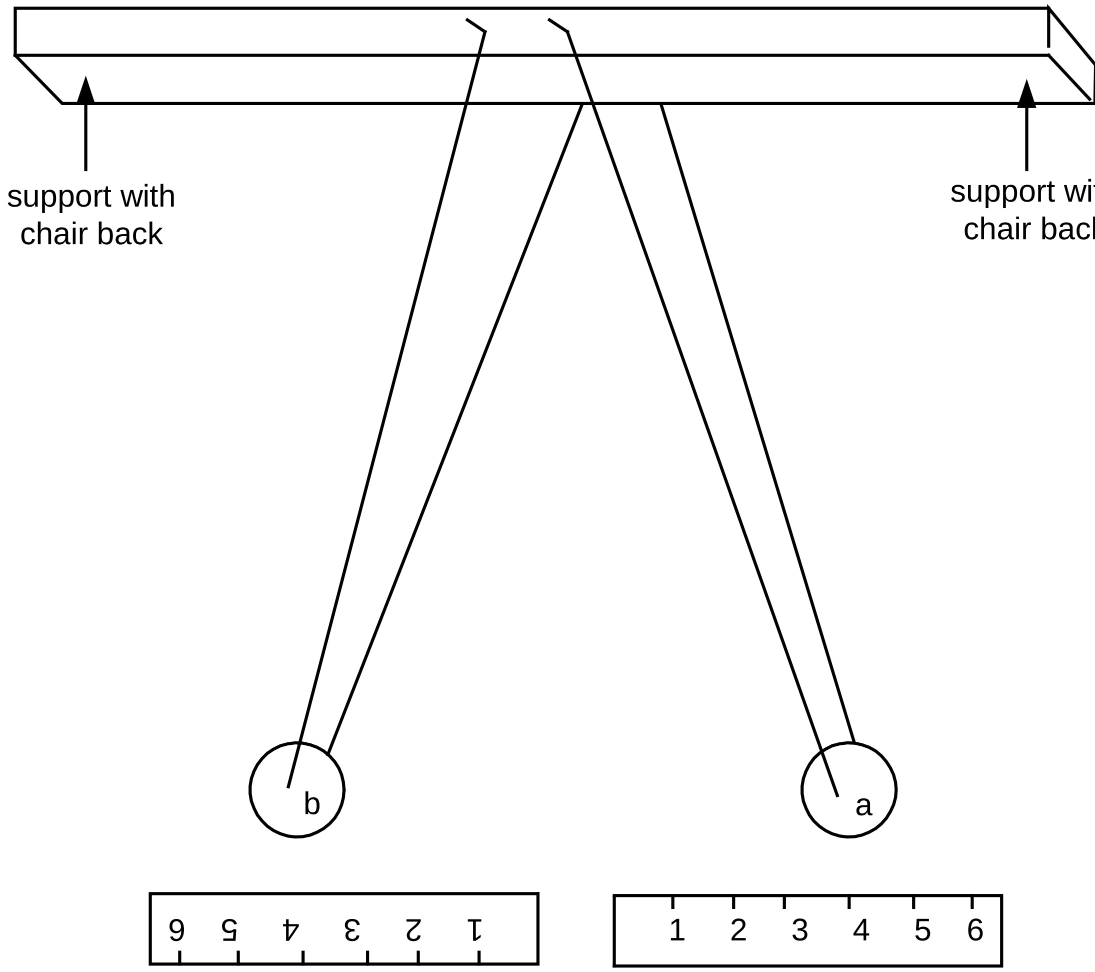
Pull back ball **a** about 4 inches from its rest position, and let it go. This is the same situation as in Example 2 where one ball is initially at rest and the other is hitting it. The velocity of ball **a** when it hits ball **b** will be proportional to the amount you pull it back. Pulling it back to 6 inches will make its velocity on collision be 50% greater than if it's pulled to 4.



When ball **a** hits **b**, does ball **a** stop as in Example 2? Does ball **b** swing out about as far from rest as ball **a** started from? If so, then ball **b** received the same velocity that **a** had, as in Example 2.

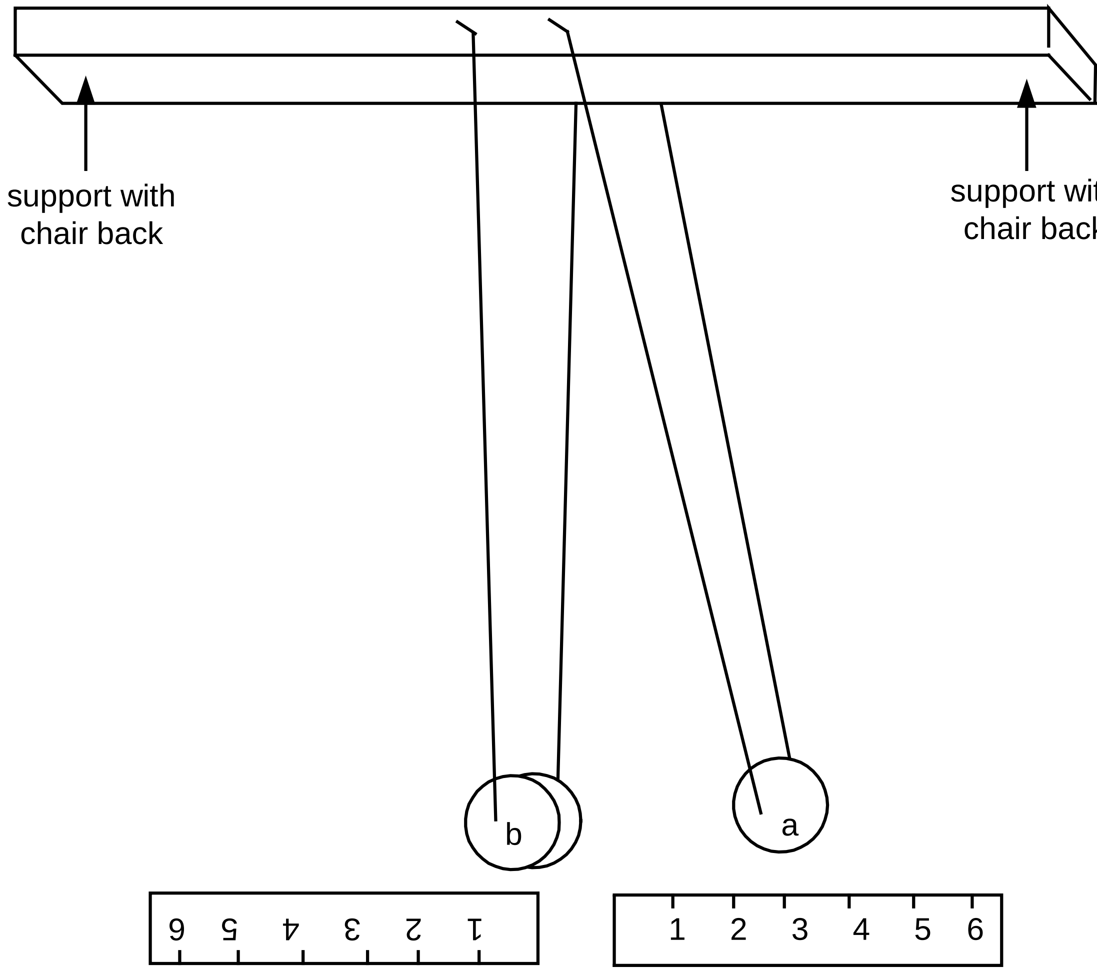
Experiment 2

Pull back both ball **a** and ball **b** the same amount and let them go. Do they act like the balls in Example 1?



Experiment 3

Change ball **b** to a double ball. That is, put two balls side-by-side on the string and put a drop of glue between them so they stay together. Ball **b** now has twice the mass of ball **a**, as in Example 3. Pull back just ball **a** and release it. See if the system acts the same as in Example 3. That is, does ball **b** receive two-thirds the velocity of ball **a** (as measured by how far it swings to the left)? Does ball **a** swing back one-third as far as it was released from? You have just one chance to make the measurement after the first impact.



Experiment 4

Pull ball **b** back half as far as you pull ball **a**, and release them. That way ball **b** will have half the speed on impact that ball **a** does, as in Example 4. Do the results agree with Example 4? Note that same impact is repeated over and over, so you'll have plenty of time to make your measurement.

Problem 1

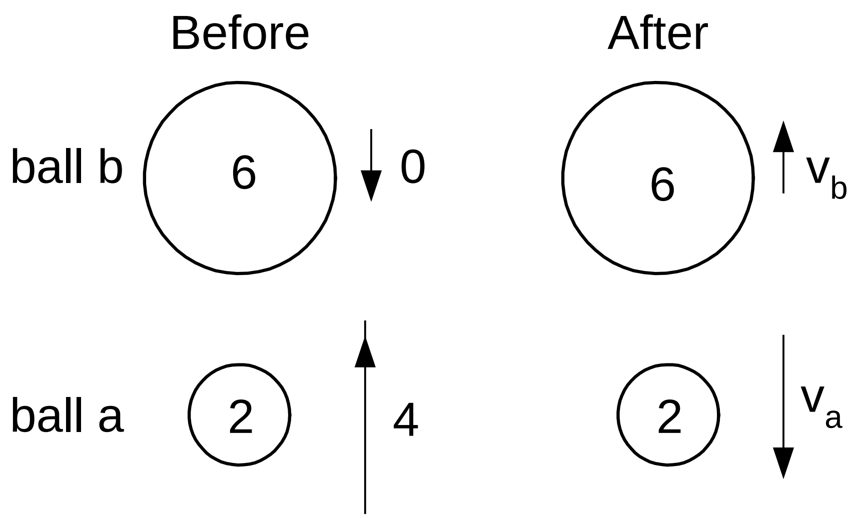
Show that the total energy in Example 3 is the same before and after the collision. Show that the total momentum is the same before and after the collision.

Problem 2

Choose a new frame of reference for Example 3 so before the collision ball **a** appears to be stopped and ball **b** is moving toward it. For this new frame of reference, what are the velocities after the collision? Show that both energy and momentum are conserved in the new frame of reference.

Problem 3

Ball **b** has three times the mass of ball **a**. If ball **a** strikes ball **b** (at rest) with velocity 4, find the velocities v_a and v_b after the collision.



Problem 4

Carry out the algebraic steps to prove the result in Appendix A.

Appendix A
Derivation of "Conservation of Momentum" Law

In a "colliding balls" experiment the energy must be conserved. That is,

$$\tfrac{1}{2} m_a (u_a)^2 + \tfrac{1}{2} m_b (u_b)^2 = \tfrac{1}{2} m_a (v_a)^2 + \tfrac{1}{2} m_b (v_b)^2$$

where u are the velocities before collision and v are the velocities after collision.

If an observer is moving past the experiment with velocity x, then all the velocities in the experiment will be reduced by x. But energy must still be conserved in his frame of reference. Then from the moving frame of reference the conservation of energy says that

$$\tfrac{1}{2} m_a (u_a - x)^2 + \tfrac{1}{2} m_b (u_b - x)^2 = \tfrac{1}{2} m_a (v_a - x)^2 + \tfrac{1}{2} m_b (v_b - x)^2$$

This must hold for all values of x.

Get two equations by letting x = 0 and x = 1. Then combine these two equations to get the result that

$$m_a u_a + m_b u_b = m_a v_a + m_b v_b,$$

which is the conservation of momentum in the original experiment. Show that momentum is also conserved in the new frame of reference—that is, that

$$m_a (u_a - x) + m_b (u_b - x) = m_a (v_a - x) + m_b (v_b - x).$$

Appendix B
Finding the "After" velocities in Example 3

Conservation of energy:

$$\mathbf{9 = v_a^2 + 2\, v_b^2}$$

Conservation of momentum:

$$\mathbf{6 = 2\, v_a + 4\, v_b}$$

From this second equation, solve for v_a in terms of v_b:

$$3 = v_a + 2\, v_b$$

$$v_a = 3 - 2v_b.$$

Substitute this expression for v_a into the Energy Conservation equation:

$$9 = (3 - 2v_b)^2 + 2\, v_b^2$$

$$9 = (3 - 2v_b)\,(3 - 2v_b) + 2\, v_b^2$$

$$9 = 3(3 - 2v_b)\; - 2v_b(3 - 2v_b) + 2\, v_b^2$$

$$9 = 9 - 6v_b\; - 6v_b + 4v_b^2 + 2\, v_b^2$$

$$0 = -\,12v_b\; + 6\, v_b^2$$

$$12v_b\; = 6\, v_b^2$$

$$\mathbf{2 = v_b}$$

Substitute this value of v_b into the Energy Conservation equation:

$$6 = 2\, v_a + 4 \cdot 2$$

$$6 - 8 = 2\, v_a$$

$$-2 = 2\, v_a$$

$$\mathbf{-1 = v_a}$$